EQUIVARIANT POINCARÉ DUALITY AND TATE COHOMOLOGY FOR COMPACT LIE GROUP ACTIONS

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Abstract. In this paper we study the question how can one generalize Poincaré duality to equivariant settings, when we replace ordinary homology and cohomology with the Borel equivariant homology and cohomology.

We first deal with the case of a finite group. We construct, using homological algebra, an equivariant cohomology theory denoted DH_G^* with two properties:

1) There is an equivariant Poincaré duality:

$$DH_G^k(M) \xrightarrow{\cong} H_{m-k}^G(M)$$

where M is a closed oriented smooth manifold of dimension m and G is a finite group acting smoothly and orientation preserving.

2) For every $G-CW$ complex X there is a long exact sequence:

$$... \to \hat{H}_G^{*-1}(X) \to DH_G^*(X) \to H_G^*(X) \to \hat{H}_G^*(X) \to ...$$

where $\hat{H}_G^*$ denotes equivariant Tate cohomology. This implies that equivariant Tate cohomology is the obstruction to equivariant Poincaré duality.

In the second part we give a similar picture when G is a compact Lie group. To do so we use bordism theories defined by Kreck using stratifolds which are generalization of manifolds, having also a stratified singular part. This description is geometric, unlike the one discussed before, hence it is suitable to cases when dealing with manifolds. To make this concrete, we give some computational examples.

1. Introduction

Let G be a topological group acting on a topological space X. One often looks at what is called the Borel construction:

$$EG \times_G X$$

where EG is a contractible space with a free proper G action. The (co)homology of this space is called the Borel (co)homology or just equivariant (co)homology. It is denoted by:

$$H_k^G(X) = H_k(EG \times_G X) \quad and \quad H_G^k(X) = H^k(EG \times_G X)$$

Equivariant (co)homology is a basic tool in the study of transformation groups. It has some nice properties, for example (assuming the action is "nice"):

1) If the action is free and proper then $EG \times_G X$ is homotopy equivalent to X/G so we have:

$$H_k^G(X) = H_k(X/G) \quad and \quad H_G^k(X) = H^k(X/G)$$

2) There is a fiber bundle $X \to EG \times_G X \to BG$ which enables us to make computations using the Serre spectral sequence.

3) If $f : X \to Y$ is an equivariant map which is a weak equivalence when forgetting the G action then f_* and f^* are isomorphisms.

For more details we refer to Allday and Puppe [4] and Brown [6].

Equivariant Poincaré duality:

Poincaré duality is a fundamental tool in the study of manifolds. In the equivariant setting we have a different picture. On the one hand, if the action is free and orientation preserving then equivariant Poincaré duality holds. On the other hand, when M is a point Poincaré duality does not hold:

$$H_k^G(pt) = H_k(BG) \ \ and \ \ H_G^k(pt) = H^k(BG)$$

and unless G is trivial there are $k > 0$ such that $H_k(BG) \neq 0$ while $H^{-k}(BG)$ is obviously trivial.

Remark. In this paper all groups acting are compact Lie group and (co)homology is with integral coefficients.

This rises two questions:

1) Is there any relation between the groups $H_k^G(M)$ and $H_G^{m-k}(M)$ where m is the dimension of M?

2) Can we replace H_G^* by another equivariant cohomology theory DH_G^* such that an equivariant Poincaré duality will hold:

$$H_k^G(M) \cong DH_G^{m-k}(M)$$

In this paper we answer both questions, first for G finite in the language of homological algebra and later for G compact Lie using a bordism approach (In the latter case we will have a dimension shift in the isomorphism of 2).

To answer these questions we start by discussing equivariant Tate cohomology (or just Tate cohomology). This theory was introduced by Swan [23] (for finite groups using homological algebra), and later was generalized by Adem, Cohen and Dwyer [3] and Greenlees and May [11] (for compact Lie groups using equivariant stable homotopy). In this paper we will not discuss the latter approach.

We now restrict to the case of finite groups. For simplicity, we will assume that the spaces are finite dimensional $G - CW$ complexes. Tate cohomology, denoted by $\hat{H}_G^*$, has some interesting properties:

(1) $\hat{H}_G^*(X) \equiv 0$ if and only if the action is free. Moreover, the inclusion of the singular part, that is all the points with non trivial stabilizer, induces an isomorphism on Tate cohomology.
(2) There is a natural transformation of equivariant multiplicative cohomology theories $H_G^*(X) \to \hat{H}_G^*(X)$ which is an isomorphism above the dimension of X.
(3) Tate cohomology is periodic for groups with the property that every Abelian subgroup is cyclic .

One can define using homological algebra a third term $DH_G^*(X)$ such that the following sequence is exact:

$$... \to \hat{H}_G^{*-1}(X) \to DH_G^*(X) \to H_G^*(X) \to \hat{H}_G^*(X) \to ...$$

We call this theory the backwards cohomology. I couldn't find it in the literature but it is rather straight forward.

From a transfer map argument we get the following:

(1) The Tate groups $\hat{H}^*_G$ are annihilated by the order of G.
(2) There is a natural map $H^*_G(M) \to DH^*_G(M)$ which factors through $H^*(M)$ such that both compositions are given by multiplication by the order of G (these groups need not be torsion groups).

The notation DH comes from the duality it has with equivariant cohomology:

Theorem. *23 (Equivariant Poincaré duality) Let M be a closed oriented smooth manifold of dimension m with a smooth action of a finite group G. We have the following isomorphisms:*

$$H^k_G(M) \to DH^G_{m-k}(M), \qquad DH^k_G(M) \to H^G_{m-k}(M), \qquad \hat{H}^k_G(M) \to \hat{H}^G_{m-k-1}(M).$$

This implies that $\hat{H}^*_G(M)$ is the obstruction for the "Poincaré duality map"

$$H^G_*(M) \to H^{m-*}_G(M)$$

from being an isomorphism.

Tate cohomology for compact Lie groups:

In the second part of this paper we give a similar picture for the case where G is a compact Lie group. We construct a similar sequence but in this case we use a geometric description of homology and cohomology. We use stratifolds and stratifold homology and cohomology. Stratifolds are a certain kind of stratified spaces. They were introduced by Kreck, and were used to define homology and cohomology theories (see [16]). We construct a long exact sequence involving three equivariant cohomology theories, which agrees with the long exact sequence discussed above, when the group is finite. This gives a generalization of Tate cohomology for compact Lie group actions. This generalization is geometric, unlike the ones discussed before, hence it is suitable to cases when dealing with manifolds. To make this concrete, we give some computational examples.

Organization of the paper:

Section 2 deals with equivariant Poincaré Duality for finite group actions. In section 3 we treat the case of compact Lie group actions, and give few examples. In the appendix we review some facts regarding locally finite homology, needed for section 3.

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2. Equivariant Poincaré Duality For Finite Group actions

Most of the results in this section are well known. Our contribution is presenting what we call the backwards (co)homology and its relation to the Borel and Tate (co)homology, especially, the equivariant Poincaré duality theorem.

2.1. **Group homology and cohomology.**

We begin with a brief introduction to group homology and cohomology. More can be found in Brown's book [6] for example.

Let G be a group and M a $\mathbb{Z}[G]$ module. The homology of G with coefficients in M is $H_*(G,M) = Tor_*^{\mathbb{Z}[G]}(\mathbb{Z},M)$ and the cohomology of G with coefficients in M is $H^*(G,M) = Ext^*_{\mathbb{Z}[G]}(\mathbb{Z},M)$. In order to compute, one has to choose a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$, or briefly a projective resolution, $P_\bullet^+$, then $H_*(G,M) \cong H_*(P_\bullet^+ \otimes_{\mathbb{Z}[G]} M)$ and $H^*(G,M) \cong H^*(Hom_{\mathbb{Z}[G]}(P_\bullet^+,M))$. This definition is independent of the choice of the projective resolution.

2.2. **Tate and Backwards (co)homology.**

A backwards projective resolution of $\mathbb{Z}$, or briefly a backwards resolution, is a chain complex $P_\bullet^-$:

$$0 \to P_0^- \to P_{-1}^- \to P_{-2}^- \to ...$$

consisting of projective modules together with a map $\mathbb{Z} \to P_\bullet^-$ which is a quasi isomorphism when considered as map between chain complexes. When G is finite, backwards resolutions always exist and are unique up to homotopy ([6] VI,3.5). It is shown there, that the dual of a projective resolution $P_\bullet^+$, namely $Hom_{\mathbb{Z}[G]}(P_\bullet^+,\mathbb{Z}[G])$, is a backwards resolution and every backwards resolution is obtained this way, up to an isomorphism.

Remark 1. In this section we will assume that G is finite unless stated otherwise.

Let $P_\bullet^+$ be a projective resolution and $P_\bullet^-$ a backwards resolution. By splicing together $P_\bullet^+$ and $P_\bullet^-[-1]$ (desuspension of $P_\bullet^-$) we get what is called a complete (projective) resolution

$$... \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to ...$$

where $P_k = P_k^+$ for $k \geq 0$ and $P_k = P_{k+1}^-$ for $k < 0$. The map $P_0 \to P_{-1}$ is the composition $P_0^+ \to \mathbb{Z} \to P_0^-$. From now on, we will use the notation $P_\bullet^+$, $P_\bullet^-$, $P_\bullet$ for a projective resolution, a backwards resolution and a complete resolution respectively.

Remark 2. We use the convention that the boundary operator in $P_\bullet^-[-1]$ is minus the boundary operator in $P_\bullet^-$.

Tate homology and cohomology are defined for finite groups in the same way as group homology and cohomology, using a complete resolution instead of a projective resolution. The definition is independent of the choice of the complete resolution ([6] VI,3.3). The Tate groups are denoted by $\hat{H}_*(G,M)$ and $\hat{H}^*(G,M)$.

We define the backwards homology and cohomology for finite groups as in Tate but using a backwards resolution. Again, this is independent of the choice of the backwards resolution (similar to the case of Tate). We denote the backwards groups by $DH_*(G,M)$ and $DH^*(G,M)$.

At a first look this doesn't look interesting due to the following:

Proposition 3. *There are isomorphisms which are natural in M:*
$DH_k(G,M) \to H^{-k}(G,M)$ *and* $DH^k(G,M) \to H_{-k}(G,M)$.

Proof. $DH_k(G,M) \cong H_k(P_\bullet^- \otimes_{\mathbb{Z}[G]} M)$. We may assume that $P_\bullet^-$ is the dual of some projective resolution $P_\bullet^+$ so $P_\bullet^- \otimes_{\mathbb{Z}[G]} M \cong Hom_{\mathbb{Z}[G]}(P_{-\bullet}^+,\mathbb{Z}[G]) \otimes_{\mathbb{Z}[G]} M$, which is naturally isomorphic, by the duality theorem ([6] I,8.3), to $Hom_{\mathbb{Z}[G]}(P_{-\bullet}^+,M)$.

Thus $DH_k(G,M) \cong H_k(Hom_{\mathbb{Z}[G]}(P^+_{-\bullet}, M)) \cong H^{-k}(G,M)$. The other statement is proved in a similar way. □

The following is proved by a standard homological algebra argument:

Proposition 4. *A short exact sequence of $\mathbb{Z}[G]$ modules:*

$$0 \to M \to M' \to M'' \to 0$$

gives rise to a long exact sequence in homology:

$$... \to H_k(G,M) \to H_k(G,M') \to H_k(G,M'') \to H_{k-1}(G,M) \to ...$$

and similarly for $\hat{H}$ and DH both in homology and cohomology.

Since $0 \to P^-_\bullet[-1] \to P_\bullet \to P^+_\bullet \to 0$ is exact we have the following:

Proposition 5. *The following are exact:*

$$... \to DH_{k+1}(G,M) \to \hat{H}_k(G,M) \to H_k(G,M) \to DH_k(G,M) \to ...$$

$$... \to \hat{H}^{k-1}(G,M) \to DH^k(G,M) \to H^k(G,M) \to \hat{H}^k(G,M) \to ...$$

for every finite group G and $\mathbb{Z}[G]$ module M.

Remark 6. The boundary maps $H_k(G,M) \to DH_k(G,M)$ and $DH^k(G,M) \to H^k(G,M)$ are induced by the chain map $P^+_\bullet \to P^-_\bullet$.

The only interesting case is when $k = 0$ since in all other cases the maps are isomorphisms or trivial. This gives us the following:

Corollary 7. *The following is exact:*

$$0 \to \hat{H}^{-1}(G,M) \to H_0(G,M) \to H^0(G,M) \to \hat{H}^0(G,M) \to 0$$

Proof. This follows from the fact that $DH^1(G,M)$ and $H^{-1}(G,M)$ vanish, and the isomorphism $DH^0(G,M) \to H_0(G,M)$. □

2.3. **(Co)homology with coefficients in a chain complex.**

There is a straightforward generalization of all those theories to the case where M is a chain complex. We recall the basic operations between chain complexes, mainly for the careful treatment of signs.

The tensor product of chain complexes $C_\bullet$ and $D_\bullet$ is the chain complex:

$$(C_\bullet \otimes D_\bullet)_n = \bigoplus_{k+l=n} C_k \otimes D_l$$

where the differential is given by:

$$\partial(c \otimes d) = \partial c \otimes d + (-1)^{|c|}(c \otimes \partial d).$$

The Hom (cochain) complex of a chain $C_\bullet$ and a cochain complex $D^\bullet$ is the cochain complex:

$$Hom(C_\bullet, D^\bullet)_n = \bigoplus_{k+l=n} Hom(C_k, D^l)$$

where the differential is given by:

$$\delta\varphi = \delta_D \circ \varphi + (-1)^{|\varphi|+1}\varphi \circ \partial_C.$$

Remark 8. There is no essential difference between chain complexes and cochain complexes, and by re-indexing we can switch between the two.

We have the following:

Proposition 9. *([6] I,0 ex. 6) Let $A_\bullet$ and $B_\bullet$ be complexes of $\mathbb{Z}[G]$ modules, and $C_\bullet$ a complex of $\mathbb{Z}$ modules. There is a natural isomorphism:*
$Hom_{\mathbb{Z}}(A_\bullet \otimes_{\mathbb{Z}[G]} B_\bullet, C_\bullet) \to Hom_{\mathbb{Z}[G]}(A_\bullet, Hom_{\mathbb{Z}}(B_\bullet, C_\bullet))$ which is a chain map.

We call this map the adjunction map.

Remark 10. Here $Hom_{\mathbb{Z}}(C_\bullet, D^\bullet) = \Pi_{k+l=n} Hom(C_k, D^l)$ and not as mentioned before. We will only use the adjunction map when there is no difference between the direct sum and the direct product.

The following is proved by a simple check of signs:

Proposition 11. *Let R be a ring and $A_\bullet$ and $B_\bullet$ be complexes of R modules. If $A_\bullet$ consists of finitely generated projective modules then there is a natural isomorphism $\varphi : Hom_R(A_\bullet, R) \otimes B_\bullet \to Hom_R(A_\bullet, B_\bullet)$ given by $\varphi(f \otimes b)(a) = (-1)^{|f||b|} f(a) \cdot b$ which is a chain map. We also call this map the duality map.*

Now, it is easy to define the homology (cohomology) of a group G with coefficients in a chain (cochain) complex $M_\bullet$ ($M^\bullet$) to be the homology of the chain (cochain) complex $P_\bullet^+ \otimes_{\mathbb{Z}[G]} M_\bullet$ ($Hom_{\mathbb{Z}[G]}(P_\bullet^+, M^\bullet)$) where $P_\bullet^+$ is a projective resolution. We denote it by $H_*(G, M_\bullet)$ ($H^*(G, M^\bullet)$). In a similar way we define the Tate and backwards homology and cohomology with coefficients in a chain/cochain complex. The analogs of propositions 4 and 5 and remark 6 are also true in this case.

All the theories are multiplicative (see [6] VI,5), and the natural transformations respect this product.

2.4. **equivariant (co)homology.**

We restrict now to the case where $M_\bullet$ ($M^\bullet$) are the cellular chain (cochain) complex of a $G-CW$ complex X which we denote by $C_\bullet(X)$ ($C^\bullet(X)$). We denote $H_*(G, C_\bullet(X))$ by $H_*^G(X)$ ($H^*(G, C^\bullet(X))$ by $H_G^*(X)$) and call it the equivariant homology (cohomology) of X. This abuse of notation is explained in the following, which is standard:

Proposition 12. *There are multiplicative natural isomorphisms*

$$H_*(EG \times_G X) \cong H_*^G(X) \text{ and } H^*(EG \times_G X) \cong H_G^*(X).$$

In a similar way, for a finite group G, equivariant Tate homology and cohomology of a $G-CW$ complex are defined and denoted by $\hat{H}_*^G(X)$ and $\hat{H}_G^*(X)$ and the backwards homology and cohomology of a $G-CW$ complex which we denote by $DH_*^G(X)$ and $DH_G^*(X)$. As before we have:

Theorem 13. *The following are exact:*

$$... \to DH_{k+1}^G(X) \to \hat{H}_k^G(X) \to H_k^G(X) \to DH_k^G(X) \to ...$$

$$... \to \hat{H}_G^{k-1}(X) \to DH_G^k(X) \to H_G^k(X) \to \hat{H}_G^k(X) \to ...$$

for every finite group G and every $G-CW$ complex X.

For a subgroup (of finite index) $G' \leq G$ there are also maps going in the "wrong" direction called the transfer map, for example $H^*_{G'}(X) \to H^*_G(X)$. The composition $H^*_G(X) \to H^*_{G'}(X) \to H^*_G(X)$ is equal to $\alpha \mapsto |G : G'| \cdot \alpha$. Similar results hold for Tate and backwards cohomology. Since $\hat{H}^*_{\{1\}}(X)$ is trivial for finite dimensional $G - CW$ complexes we get the following:

Proposition 14. *Let X be a finite dimensional $G - CW$ complex, then there is a natural map $H^k_G(X) \to DH^k_G(X)$ such that the compositions $H^k_G(X) \to DH^k_G(X) \to H^k_G(X)$ and $DH^k_G(X) \to H^k_G(X) \to DH^k_G(X)$ are given by multiplication by $|G|$ and factor through $H^k_{\{1\}}(X) \cong H^k(X)$.*

2.5. **The spectral sequences.**

Remark 15. In this section we use the language of [7] for spectral sequences.

There are two spectral sequences associated to the different filtrations of a double complex. A careful analysis is needed since some of those spectral sequences are not bounded.

In the case of equivariant homology and cohomology the two double complexes are first quadrant and so the spectral sequences are bounded. Therefore we have the following ([6]VII,5.3,5.6):

Theorem 16. *The following spectral sequences strongly converge:*
$E^2_{pq} = H_p(G, H_q(X)) \Longrightarrow H^G_{p+q}(X)$ *and* $E^{pq}_2 = H^p(G, H^q(X)) \Longrightarrow H^{p+q}_G(X)$
$E^1_{pq} = H_q(G, C_p(X)) \Longrightarrow H^G_{p+q}(X)$ *and* $E^{pq}_1 = H^q(G, C^p(X)) \Longrightarrow H^{p+q}_G(X)$

In the case of Tate cohomology and backwards cohomology the first filtration is regular and hence strongly convergent ([7] XV,4.1):

Theorem 17. *The following spectral sequences strongly converge:*
$E^{pq}_2 = \hat{H}^p(G, H^q(X)) \Longrightarrow \hat{H}^{p+q}_G(X)$ *and* $E^{pq}_2 = DH^p(G, H^q(X)) \Longrightarrow DH^{p+q}_G(X)$.

In case X is finite dimensional the spectral sequences associated to both filtrations are bounded and therefore strongly converge:

Theorem 18. *Let X be a finite dimensional $G - CW$ complex. The following spectral sequences strongly converge:*
$E^{pq}_1 = \hat{H}^q(G, C^p(X)) \Longrightarrow \hat{H}^{p+q}_G(X)$ *and* $E^{pq}_1 = DH^q(G, C^p(X)) \Longrightarrow DH^{p+q}_G(X)$
$E^1_{pq} = \hat{H}_q(G, C_p(X)) \Longrightarrow \hat{H}^G_{p+q}(X)$ *and* $E^1_{pq} = DH_q(G, C_p(X)) \Longrightarrow DH^G_{p+q}(X)$
$E^2_{pq} = \hat{H}_p(G, H_q(X)) \Longrightarrow \hat{H}^G_{p+q}(X)$ *and* $E^2_{pq} = DH_p(G, H_q(X)) \Longrightarrow DH^G_{p+q}(X)$

Here are some corollaries of the above:

Corollary 19. *([6] VII,7.3) Let $f : X \to Y$ be an equivariant map between two $G - CW$ complexes which is a weak equivalence then the maps $H^G_n(X) \to H^G_n(Y)$ ($H^n_G(Y) \to H^n_G(X)$) are isomorphisms. Similar results hold for Tate and backwards homology and cohomology (for homology we have to assume that X and Y are finite dimensional $G - CW$ complexes). In particular, for f the identity map, this implies that the above theories are independent of the cellular structure (for $\hat{H}^G_*(X)$ and $DH^G_*(X)$ we have to assume that X is finite dimensional).*

Proposition 20. *([6] VII,7.3) Let X be finite dimensional and Y its singular part, that is all points with non trivial stabilizer, then $\hat{H}^*_G(X) \to \hat{H}^*_G(Y)$ is an isomorphism. In particular, if X is free, $\hat{H}^*_G(X)$ is trivial (hence $DH^*_G(X) \cong H^*_G(X) \cong H^*(X/G)$).*

Corollary 21. *Let X be a finite dimensional $G-CW$ complex then an element $\alpha \in DH_G^k(X)$ is in the kernel of the map $DH_G^k(X) \to H_G^k(X)$ if and only if it is in the kernel of the map $f^* : DH_G^k(X) \to DH_G^k(Y)$ for every equivariant map $f : Y \to X$ where Y is a finite dimensional $G-CW$ complex with a free G action.*

Proof. Let EG_k be a highly connected, finite dimensional $G-CW$ complex with a free G action. In this case we have:

$$\begin{array}{ccccccc}
\hat{H}_G^{k-1}(X) & \to & DH_G^k(X) & \to & H_G^k(X) & \to & \hat{H}_G^k(X) \\
 & & \downarrow & & \downarrow\cong & & \\
0 & \to & DH_G^k(X \times EG_k) & \xrightarrow{\cong} & H_G^k(X \times EG_k) & \to & 0
\end{array}$$

(we use proposition 20 in the second line). We get that:

$$ker\left(DH_G^k(X) \to H_G^k(X)\right) = ker\left(DH_G^k(X) \to DH_G^k(X \times EG_k)\right).$$

This completes the proposition since for any Y having a free action, we can find such EG_k with a map $Y \to EG_k$, so f factors through $X \times EG_k$. □

2.6. **Equivariant Poincaré duality.**

Let M be a closed oriented smooth manifold of dimension m with a smooth and orientation preserving action of a finite group G. There is an equivariant triangulation of M ([15]) which gives M a structure of a $G-CW$ complex. Let $\sigma_M \in C_m(M)$ be a fundamental cycle. For any $g \in G$ we know that $g \cdot \sigma_M$ is also a fundamental cycle since the action is orientation preserving. Since $C_{m+1}(M) = 0$ we deduce that $g \cdot \sigma_M = \sigma_M$.

Proposition 22. *The map $T : C^k(M) \to C_{m-k}(M)$ defined by $T(\varphi) = \varphi \cap \sigma_M$ is a $\mathbb{Z}[G]$ chain map. T induces an isomorphism $DH_G^k(M) \to H_{m-k}^G(M)$.*

Proof. Defining the cap product on cellular chains requires a choice of a cellular approximation of the diagonal map $\Delta : M \to M \times M$. We can choose it to be equivariant. Since σ_M is invariant it is an easy check that T is actually a $\mathbb{Z}[G]$ chain map. By Poincaré duality T is a quasi isomorphism.

Choose a projective resolution P_*^+ of finite type and let P_*^- be the dual backwards resolution. By the duality isomorphism we have:

$$Hom_{\mathbb{Z}[G]}(P_*^-, C^*(M)) \cong Hom_{\mathbb{Z}[G]}(P_*^-, \mathbb{Z}[G]) \otimes_{\mathbb{Z}[G]} C^*(M) \cong P_{-*}^+ \otimes_{\mathbb{Z}[G]} C^*(M).$$

We also have the map:

$$Id \otimes T : P_{-*}^+ \otimes_{\mathbb{Z}[G]} C^*(M) \to P_{-*}^+ \otimes_{\mathbb{Z}[G]} C_{m-*}(M)$$

which is a quasi isomorphism by a spectral sequence argument. The composition induces an isomorphism $DH_G^k(M) \to H_{m-k}^G(M)$. □

The following theorem is proved similarly:

Theorem 23. *Let M be a smooth oriented closed manifold of dimension m with a smooth and orientation preserving action of a finite group G. We have the following isomorphisms:*

$$H_G^k(M) \to DH_{m-k}^G(M), \qquad DH_G^k(M) \to H_{m-k}^G(M), \qquad \hat{H}_G^k(M) \to \hat{H}_{m-k-1}^G(M).$$

Moreover, the following diagram commutes:

$$\begin{array}{ccccccccc}
\ldots \to & \hat{H}_G^{k-1}(M) & \to & DH_G^k(M) & \to & H_G^k(M) & \to & \hat{H}_G^k(M) & \to \ldots \\
& PD_M \downarrow & & PD_M \downarrow & (1) & PD_M \downarrow & & PD_M \downarrow & \\
\ldots \to & \hat{H}^G_{m-k}(M) & \to & H^G_{m-k}(M) & \to & DH^G_{m-k}(M) & \to & \hat{H}^G_{m-k-1}(M) & \to \ldots
\end{array}$$

The composition of the isomorphism $H^G_{m-k}(M) \xrightarrow{PD_M^{-1}} DH_G^k(M)$ with the map $DH_G^k(M) \to H_G^k(M)$ in (1) gives a map $H^G_{m-k}(M) \to H_G^k(M)$. This map is not an isomorphism in general, the Tate groups are the obstruction for that. Note that the Tate groups are trivial for every k if and only if G acts freely on M. If we were able compute this map we would have been able to compute $\hat{H}_G^*(M)$ up to extension.

Remark 24. Let G^{ad} be G together with the adjoint representation, then the product $G^{ad} \times G^{ad} \to G^{ad}$ is equivariant, giving us a product:

$$DH_k^G\left(G^{ad}\right) \otimes DH_l^G\left(G^{ad}\right) \to DH_{k+l}^G\left(G^{ad}\right)$$

using the duality we get a map in equivariant cohomology:

$$H_G^k\left(G^{ad}\right) \otimes H_G^l\left(G^{ad}\right) \to H_G^{k+l}\left(G^{ad}\right)$$

This product is different from the cup product. This can be seen by noticing that this product does not restrict to a product on the components of the space $EG \times_G G^{ad}$.

Using the fact that $EG \times_G G^{ad}$ is homotopy equivalent to LBG, the free loop space of BG, this gives a product:

$$H^k(LBG) \otimes H^l(LBG) \to H^{k+l}(LBG)$$

This product is related to the one defined in [13].

In the next part, by a similar argument, we will see that one can define this product for compact Lie groups, but there we will get a dimension shift:

$$H^k(LBG) \otimes H^l(LBG) \to H^{k+l-d}(LBG)$$

3. Equivariant Poincaré Duality And Tate Cohomology For Compact Lie Group actions

In this section we present a similar picture to the one we had before, but now the group acting will be compact Lie instead of finite. In order to do so we use a completely different approach, a geometric one. We use a bordism description of homology and cohomology using stratifolds. We define equivariant stratifold (co)homology theories of various types, relate them to each other and prove some fundamental properties: we relate those theories by a long exact sequence, identify them with the theories introduced in section 2 in case the group is finite and prove Poincaré duality. Furthermore, we demonstrate their potential for computation by looking at concrete examples.

This section is organized as follows:

In 3.1 we give a brief introduction to stratifolds, stratifold homology and stratifold cohomology. We also discuss locally finite stratifold homology. It will be used later in the construction of the natural isomorphisms to the theories defined before. A discussion regarding locally finite homology can be found in the appendix.

In 3.2 we discuss basic properties of smooth actions on stratifolds.

We then introduce, in pairs, the equivariant homology and their (Poincaré) dual cohomology theories and construct natural isomorphisms to the theories we discussed before in the case where G is finite. In the case of Tate cohomology, due to some technical difficulties, we could not prove naturality, but we believe that the transformation is natural:

3.3 Equivariant stratifold homology and stratifold backwards cohomology.

3.4 Equivariant stratifold cohomology and stratifold backwards homology.

3.5 Stratifold Tate cohomology and stratifold Tate homology.

We end this section in 3.6 with examples. We demonstrate how one can make computations in the case of a weighted action of S^1 on S^{2n-1}. In this case both Tate cohomology and the duality map are non trivial. On the other hand we give a large class of examples where the duality map is trivial, like in the case of Hamiltonian torus action on symplectic manifolds, and GKM manifold.

3.1. **Stratifolds, stratifold homology, stratifold cohomology and locally finite stratifold homology.**

Stratifolds are generalization of manifolds. They were introduced by Kreck [16] and used in order to define a bordism theory, denoted by SH_*, which is naturally isomorphic to singular homology. Kreck also defined a cohomology theory using stratifolds which is defined on the category of smooth oriented manifolds (without boundary but not necessarily compact). It is denoted by SH^* and is naturally isomorphic to singular cohomology.

Stratifolds.

Kreck defined stratifolds as spaces with a sheaf of functions, called the smooth functions, fulfilling certain properties, but for our purpose the following definition is enough (these stratifolds are also called p-stratifolds):

A stratifold is a pair consisting of a topological space and a subsheaf of the sheaf of real continuous functions, which is constructed inductively in a similar way to the way we construct CW complexes. We start with a discrete set of points denoted by X^0 and define inductively the set of smooth functions which in the case of X^0 are all real functions.

Suppose X^{n-1} together with a smooth set of functions is given. Let W be an n dimensional smooth manifold "the n strata" with boundary and a collar c, and f a continuous proper map from the boundary of W to X^{n-1}. We require that f is smooth, which means that its composition with every smooth map from X^{n-1} is smooth. Define $X^n = X^{n-1} \cup_f W$. The smooth maps on X^n are defined to be those maps $g : X^n \to \mathbb{R}$ which are smooth when restricted to X^{n-1} and to W and such that for some $0 < \delta$ we have $gc(x,t) = gf(x)$ for all $x \in \partial W$ and $t < \delta$.

Among the examples of stratifolds are manifolds, real algebraic varieties [12], the cone over a stratifold and the product of two stratifolds.

We can also define stratifolds with boundary which are analogous to manifolds with boundary. A main difference is that every stratifold is the boundary of its cone, which is a stratifold with boundary.

Given two stratifolds with boundary (T', S') and (T'', S'') and an isomorphism $f : S' \to S''$ there is a well defined stratifold structure on the space $T' \cup_f T''$ which is called the gluing. On the other hand, given a smooth map $g : T \to \mathbb{R}$ such that there is a neighborhood of 0 which consists only of regular values then the

preimages $g^{-1}((-\infty, 0]) = T'$ and $g^{-1}([0, \infty)) = T''$ are stratifolds with boundary and T is isomorphic to the gluing $T' \cup_{Id} T''$.

To obtain singular homology we specialize our stratifolds in the following way: We use compact stratifolds, require that their top stratum will be oriented and the codimension one stratum will be empty.

Remark. Regarding regularity, a condition often required, see a note by Kreck [18].

Stratifold homology.

Stratifold homology was defined by Kreck in [16]. We will describe here a variant of this theory called parametrized stratifold homology, which is naturally isomorphic to it for CW complexes. In this paper we will refer to parametrized stratifold homology just as stratifold homology and use the same notation for it - SH_*. It is naturally isomorphic to integral homology and gives a geometric perspective to it.

Definition 25. Let X be a topological space and $k \geq 0$, define $SH_k(X)$ to be $\{g : S \to X\} / \sim$ i.e., bordism classes of maps $g : S \to X$ where S is a compact oriented stratifold of dimension k and g is a continuous map. We often denote the class $[g : S \to X]$ by $[S, g]$ or by $[S \to X]$. $SH_k(X)$ has a natural structure of an Abelian group, where addition is given by disjoint union of maps and the inverse is given by reversing the orientation. If $f : X \to Y$ is a continuous map then we can define an induced map by composition $f_* : SH_k(X) \to SH_k(Y)$.

One constructs a boundary operator and proves the following:

Theorem 26. *(Mayer-Vietoris) The following sequence is exact:*

$$... \to SH_k(U \cap V) \to SH_k(U) \oplus SH_k(V) \to SH_k(U \cup V) \xrightarrow{\partial} SH_{k-1}(U \cap V) \to ...$$

where, as usual, the first map is induced by inclusions and the second is the difference of the maps induced by inclusions.

SH_* with the boundary operator is a multiplicative homology theory. The following can be found in [16]:

Theorem 27. *There is a natural isomorphism of multiplicative homology theories* $\Phi : SH_* \to H_*$ *which is given by* $\Phi_k([S, f]) = f_*([S])$ *where* $[S] \in H_k(S, \mathbb{Z})$ *is the fundamental class of* S.

Stratifold cohomology.

Similarly, Kreck defined stratifold cohomology. This cohomology theory, denoted by SH^*, is defined on the category of oriented manifolds. It is naturally isomorphic to integral cohomology. Poincaré duality for a closed oriented smooth manifold M of dimension m is given by $SH^k(M) \xrightarrow{\cong} SH_{m-k}(M)$ which is trivial.

Definition 28. Let M be a smooth oriented manifold of dimension m (not necessarily compact) and $k \geq 0$, define $SH^k(M)$ to be $\{g : S \to M\} / \sim$ i.e., bordism classes of maps $g : S \to M$ where S is an oriented stratifold of dimension $m - k$ and g is a smooth proper map. $SH^k(M)$ has a natural structure of an Abelian group, where addition is given by disjoint union of maps and the inverse is given by reversing the orientation. If $f : N \to M$ is a smooth map than we can define an induced map $f^* : SH^k(M) \to SH^k(N)$ by pullback (after making f transversal to g). See [16] for details on how do we orient this pullback.

Locally finite stratifold homology.

We give a short description of locally finite stratifold homology. It is used later on to construct natural isomorphisms between the geometrically defined cohomology theories and the algebraically defined cohomology theories.

Locally finite stratifold homology is denoted by SH_*^{lf}. It is defined the same way as SH_* with the exception that instead of using compact stratifolds we look at bordism classes of proper maps. Since the composition of proper maps is proper, induced maps are defined for proper maps.

Since a continuous map from a compact space to a Hausdorff space is proper there is a natural transformation $SH_k(X) \to SH_k^{lf}(X)$. We have the following ([25], 3.42):

Proposition 29. *There is a natural transformation $\Phi^{lf} : SH_*^{lf}(X) \to H_*^{lf}(X)$ which is an isomorphism for all strongly locally finite CW complexes.*

See appendix 1 for the definition and properties of locally finite homology - $H_*^{lf}(X)$ and strongly locally finite CW complexes.

Here is a version of Poincaré duality:

Theorem 30. *Let M be a smooth oriented manifold of dimension m then there is an isomorphism $PD_M : SH^k(M) \to SH_{m-k}^{lf}(M)$.*

It is used in order to prove the following ([25], 4.11):

Theorem 31. *The composition $SH^k(M) \to SH_{m-k}^{lf}(M) \to H_{m-k}^{lf}(M) \to H^k(M)$ is a natural isomorphism of multiplicative cohomology theories, where the maps $SH^k(M) \to SH_{m-k}^{lf}(M)$ and $H_{m-k}^{lf}(M) \to H^k(M)$ are the Poincaré duality maps.*

3.2. **Smooth actions on stratifolds.**

Let S be a stratifold with an action of a compact Lie group G. We say that the action is smooth if the map $\rho : G \times S \to S$ is smooth. In this case all maps $\rho_g : S \to S$ given by $\rho_g(x) = \rho(g, x)$, are diffeomorphisms. This implies that G acts on each strata separately. We would like to require a bit more. Assume that $S = M_0 \cup_{\partial M_1} M_1 ... \cup_{\partial M_n} M_n$ then we want that the action could be extended to every manifold with boundary $(M_k, \partial M_k)$ separately and that the gluing maps will be equivariant. By abuse of notation, we say that an action is "orientation preserving" if the induced action of G/G_0 on S/G_0 is orientation preserving, where G_0 is the component of the identity.

Here is a general theorem that we will use ([5] II5.8):

Theorem 32. *Suppose X is a completely regular G space, G compact Lie, and that all orbits have type G/H. Then the orbit map $X \to X/G$ is the projection in a fiber bundle with fiber G/H and structure group $N(H)/H$ (acting by right translation on G/H). Conversely every such bundle comes from such an action.*

In particular, in the case of $H = \{1\}$, the theorem states that if the action is free then the map $X \to X/G$ is a principal G bundle and every principal G bundle comes from a free action on the total space X. For this reason, we will assume that all the spaces are completely regular. This is not restrictive, since CW complexes and manifolds are completely regular.

Here are some properties of stratifolds with a free action and their quotients. The proofs are straightforward:

Lemma 33. *([25] 6.3+6.4) Let S be a stratifold of dimension k (with boundary) with a smooth G action.*

1. *If the action is free then S/G has a unique structure of a stratifold of dimension $k - dim(G)$ (with boundary), such that the map $\pi : S \to S/G$ is smooth.*
2. *If S is compact then so is S/G.*
3. *If S is oriented and the action is orientation preserving then there is a natural orientation for S/G.*
4. *If the action is free then $\pi : S \to S/G$ is a principal G bundle.*

If S be a stratifold of dimension k and $\widetilde{S} \to S$ a principal G-bundle then:

1. *There is a unique stratifold structure on $\widetilde{S}$ of dimension $k + dim(G)$ such that the quotient is smooth.*
2. *If S and G are compact then so is $\widetilde{S}$.*
3. *If S is oriented there is a way to give $\widetilde{S}$ an orientation such that the action will be orientation preserving and the map $\widetilde{S}/G \to S$ will be orientation preserving using the convention above.*

3.3. **Equivariant stratifold homology and its dual stratifold backwards cohomology.**

We now describe equivariant stratifold homology and stratifold backwards cohomology and construct natural isomorphisms to the corresponding theories described before.

Equivariant stratifold homology.

Equivariant stratifold homology, denoted by SH_*^G, is an equivariant homology theory defined by Kreck in [17]. Here G is a compact Lie group. SH_*^G is naturally isomorphic to the Borel homology after a shift by the dimension of G

$$SH_*^G(X) \cong H_{*-dim(G)}(EG \times_G X)$$

(assuming X is completely regular).

Definition 34. Let G be a compact Lie group, X a G space and $k \geq 0$, define $SH_k^G(X) = \{g : S \to X\}_G / \sim$ i.e., bordism classes of equivariant maps $g : S \to X$ where:

- S is a compact oriented stratifold of dimension k with a G action.
- The action of G on S is free, smooth and orientation preserving.
- g is a continuous equivariant map.
- The bordism relation has to fulfill the same properties as S does. In particular the action on the cobordism should be free and extend the action on the boundary.

As in the non equivariant case, $SH_k^G(X)$ has a natural structure of an Abelian group, one can define induced maps by composition, triples, boundary maps and prove a Mayer Vietoris theorem.

If H is a closed subgroup of G then every G space has a natural structure of an H space. We define the transfer map $tr_H^G : SH_k^G(X) \to SH_k^H(X)$ by restriction.

The cross product $\times : SH_k^G(X) \otimes SH_l^{G'}(Y) \to SH_{k+l}^{G\times G'}(X \times Y)$ is defined by

$$[g_1 : S \to X] \times [g_2 : T \to Y] = (-1)^{dim(G)(l-dim(G'))}[g_1 \times g_2 : S \times T \to X \times Y].$$

This product is bilinear and natural. If $G = G'$ we can compose this product with the transfer map with respect to the diagonal map $\Delta : G \to G \times G$ and get a cross product $\times : SH_k^G(X) \otimes SH_l^G(Y) \to SH_{k+l}^G(X \times Y)$.
SH_*^G is a multiplicative equivariant homology theory.

A natural isomorphism between SH_*^G and $H_{*-dim(G)}(EG \times_G -)$.
We would like to construct a natural isomorphism $SH_*^G \to H_{*-dim(G)}(EG \times_G -)$.

Lemma 35. *Let X be a G space, the projection $\pi_X : EG \times X \to X$ induces an isomorphism $\pi_{X*} : SH_*^G(EG \times X) \to SH_*^G(X)$.*

Proof. The inverse of π_{X*} is given by $[g : S \to X] \mapsto [f \times g : S \to EG \times X]$ where $f : S \to EG$ is the classifying map defined by the universal property of EG and the fact that the action on S is free. f is unique up to homotopy thus the map is well defined. □

Lemma 36. *Let X be a space with a free G action, then there is a natural isomorphism $SH_*^G(X) \to SH_{*-dim(G)}(X/G)$.*

Proof. We define this map by $[S \to X] \mapsto [S/G \to X/G]$. This is well defined by lemma 33 and it has an inverse $[S \to X/G] \mapsto [\widetilde{S} \to X]$ which is also well defined by the same lemma and theorem 32. □

Denote by Φ_0^G the composition of natural isomorphisms:

$$SH_*^G(X) \to SH_*^G(EG \times X) \to SH_{*-dim(G)}(EG \times_G X) \to H_{*-dim(G)}(EG \times_G X)$$

(the second map is an isomorphism since the action is free). The following is straightforward:

Theorem 37. *Φ_0^G is a natural isomorphism of equivariant homology theories and commutes with the transfer.*

Remark. The sign we introduced for the cross product makes Φ_0^G strictly commute with $\times$.

When G is finite and X is a $G - CW$ complex we can compose Φ_0^G with the isomorphism $H_*(EG \times_G X) \to H_*^G(X)$. Denote the composition $SH_*^G(X) \to H_*^G(X)$ by Φ^G, then we have the following:

Theorem 38. *Φ^G is a natural isomorphism of equivariant homology theories and commutes with the transfer.*

Proof. The map $H_*(EG \times_G X) \to H_*^G(X)$ is a natural isomorphism and commutes with the boundary (see [4] 1.2.8). It is not hard to see that it commutes with tr_H^G and the cross product. □

Locally finite equivariant stratifold homology.
Locally finite equivariant stratifold homology, denoted by $SH_*^{lf,G}$, is defined in a similar way to SH_*^G, but instead of compact stratifolds we use proper maps from arbitrary stratifolds. We will not get into all the details, since they are similar to what we had before, we just stress the differences.

Since induced maps are defined only for proper maps we use the following version:

Proposition 39. *Let X be a G space and EG_r an r connected closed oriented manifold with an orientation preserving free G action. The map $\pi_X : EG_r \times X \to X$ is proper and induces an isomorphism $\pi_{X*} : SH_k^{lf,G}(EG_r \times X) \to SH_k^{lf,G}(X)$ for $k < r-1$.*

Proof. We cannot follow the same proof we used before since EG is not compact thus the projection $EG \times X \to X$ is not proper. Therefore we approximate EG by EG_r for r big enough. The inverse of π_{X*} is given by:

$$[g : S \to X] \mapsto [f \times g : S \to EG_r \times X]$$

where $f : S \to EG_r$ is the classifying map defined by the universal property of EG_r and the fact that the action on S is free and S has the homotopy type of a CW complex of dimension $\leq k$. f is unique up to homotopy thus the map is well defined. Note that since g is proper so is $f \times g$. □

Proposition 40. *Let X be a space with a free G action, then there is a natural isomorphism $SH_k^{lf,G}(X) \to SH_{k-dim(G)}^{lf}(X/G)$.*

Proof. This is proved as for SH_*^G, one just have to note that since G is compact the map $[g : S \to X]$ is proper if and only if the map $[g/G : S/G \to X/G]$ is. □

Corollary 41. *Let X be a strongly locally finite $G-CW$ complex, there is a natural isomorphism $\Phi_r^{lf,G} : SH_k^{lf,G}(X) \to H_{k-dim(G)}^{lf}(EG_r \times_G X)$ where $k < r$. $\Phi_r^{lf,G}$ commutes with the boundary, the transfer and with the cross product.*

Proof. We define this isomorphism to be the composition:

$$SH_k^{lf,G}(X) \to SH_k^{lf,G}(EG_r \times X) \to SH_{k-dim(G)}^{lf}(EG_r \times_G X) \to H_{k-dim(G)}^{lf}(EG_r \times_G X)$$

where the second isomorphism is well defined since the action on $EG_r \times X$ is free. $\Phi_r^{lf,G}$ commutes with the boundary, the transfer and the cross product for the same reason Φ_0^G does. □

Using the locally finite chain complex $C_*^{lf}(X)$ instead of $C_*(X)$ one defines (when G is finite) $H_k^{lf,G}(X)$ to be $H_k(P_*^+ \otimes C_*^{lf}(X))$, then we prove the following:

Corollary 42. *Let G be a finite group and X a strongly locally finite $G-CW$ complex. There is a natural isomorphism $\Phi^{lf,G} : SH_k^{lf,G}(X) \to H_k^{lf,G}(X)$, which commutes with the boundary, the transfer and the cross product.*

Proof. EG_r are compact so we can identify $C_*^{lf}(EG_r)$ with $C_*(EG_r)$. Denote $P_*^+ = \varinjlim(C_*(EG_r))$ then $P_*^+ = C_*(\varinjlim(EG_r)) = C_*(EG)$ so it is a projective resolution. Since $\varinjlim$ commutes with homology and with tensor products we have:

$$\varinjlim(H_k^{lf}(EG_r \times_G X)) = H_k(\varinjlim(C_*^{lf}(EG_r \times_G X))) \cong H_k(P_*^+ \otimes C_*^{lf}(X)) = H_k^{lf,G}(X)$$

We saw before that $\Phi_r^{lf,G}$ is an isomorphism for r large enough therefore $\Phi^{lf,G}$ is also an isomorphism. $\Phi_r^{lf,G}$ commutes with boundary and with the cross product for all n and also the maps $H_k^{lf}(EG_r \times_G X) \to H_k^{lf}(EG_{r+1} \times_G X)$ which imply the same thing for the colimit (both the boundary and the cross product commute with the colimit since in order to compute them we only need the group $H_k^{lf}(EG_r \times_G X)$ for $k < r$ and those stabilize. □

Stratifold backwards cohomology.

Stratifold backwards cohomology, denoted by DSH^*_G, is an equivariant cohomology theory defined in [17] (there it is denoted SH^*_G and called equivariant stratifold cohomology). It is defined on the category of smooth oriented manifolds with a smooth, orientation preserving G action and equivariant maps between them, where G is a compact Lie group. Equivariant Poincaré duality is trivial. The name suggests that it is naturally isomorphic to backwards cohomology when G is finite.

Definition 43. Let G be a compact Lie group and M a smooth oriented manifold of dimension m with an orientation preserving smooth G action. For $k \leq m$, define $DSH^k_G(M) = \{g : S \to M\}_G / \sim$ i.e., bordism classes of equivariant maps $g : S \to M$ where:

- S is an oriented stratifold of dimension $m-k$ with a G action.
- The action of G on S is orientation preserving, smooth and free.
- g is a smooth equivariant proper map.
- The bordism relation has to fulfill the same properties as the stratifolds and the action does. In particular the action on the cobordism should be free and extend the action on the boundary.

As in the non equivariant case, $DSH^k_G(X)$ has a natural structure of an Abelian group, one can define induced maps by pullback (for equivariant transversality see [17] lemma 5), triples, boundary maps and prove a Mayer Vietoris theorem.

Restriction of the action to a closed subgroup $H \leq G$ gives a map called the restriction map $res^G_H : DSH^k_G(M) \to DSH^k_H(M)$.

The cross product $DSH^k_G(M) \otimes DSH^l_{G'}(N) \to DSH^{k+l}_{G\times G'}(M \times N)$ given by:

$$[g_1 : S \to M] \times [g_2 : T \to N] = (-1)^{(m+dim(G))l}[g_1 \times g_2 : S \times T \to M \times N].$$

This product is bilinear and natural. Again, by restriction, one defines a cross product $DSH^k_G(M) \otimes DSH^l_G(N) \to DSH^{k+l}_G(M \times N)$.

The cup product is given by $\alpha \cup \beta = \Delta^*(\alpha \times \beta)$ where $\Delta : M \to M \times M$ is the diagonal map.

DSH^*_G is a multiplicative equivariant cohomology theory. Its main property is equivariant Poincaré duality:

Theorem 44. *Let M be a closed oriented smooth manifold of dimension m with a smooth and orientation preserving G action then there is an isomorphism:*

$$PD_M : DSH^k_G(M) \to SH^G_{m-k}(M).$$

In case M is not compact we have the following version:

$$PD_M : DSH^k_G(M) \to SH^{lf,G}_{m-k}(M).$$

A natural isomorphism between DSH^*_G and DH^*_G.

We now want to construct a natural isomorphism $\Theta^D_G : DSH^*_G \to DH^*_G$. Since DH^*_G is defined only for finite groups, assume that G is a fixed finite group. The construction of Θ^D_G is simple, given what we already showed. Let M be a smooth oriented manifold of dimension m with a smooth and orientation preserving G action. Define $\Theta^D_G(M)$ to be the composition:

$$DSH^k_G(M) \to SH^{lf,G}_{m-k}(M) \to H^{lf,G}_{m-k}(M) \to DH^k_G(M).$$

This is an isomorphism of Abelian group as a composition of such isomorphisms. Proving that it is a natural isomorphism of multiplicative equivariant cohomology theories is rather technical, and this is what we will do now.

We choose $r > m - k + 1$ and let EG_r be an r connected closed oriented smooth manifold of dimension d_r with a free and orientation preserving G action.

Remark 45. From now on we will assume that all d_r are even. This way we avoid some sign problems that would have occurred otherwise.

Lemma 46. *There is a natural isomorphism:*

$$C^*(EG_r) \otimes_{\mathbb{Z}[G]} C^*(M) \xrightarrow{\times_G} C^*(EG_r \times_G M)$$

Proof. G is finite so $C^*(EG_r) \cong Hom_{\mathbb{Z}[G]}(C_*(EG_r), \mathbb{Z}[G])$ (see [6] VI3.4).

$C_*(EG_r)$ are finitely generated since EG_r is compact and projective since G acts freely on EG_r. By the duality theorem, the left side is isomorphic to:

$$\xrightarrow{\cong} Hom_{\mathbb{Z}[G]}(C_*(EG_r), C^*(M)).$$

By adjunction this is isomorphic to:

$$\xrightarrow{\cong} Hom_{\mathbb{Z}}(C_*(EG_r) \otimes_{\mathbb{Z}[G]} C_*(M), \mathbb{Z}) \xrightarrow{\cong} Hom_{\mathbb{Z}}(C_*(EG_r \times_G M), \mathbb{Z})$$

which is the right side by definition.

One might check that the composition is given by:
$\varphi \times_G \psi(e \otimes m) = (-1)^{|\varphi| \cdot |\psi|} \Sigma_{g \in G} \varphi(ge) \cdot \psi(gm)$ □

Lemma 47. *The following diagram commutes up to homotopy:*

$$\begin{array}{ccc} C^{d_r+*}(EG_r) \otimes_{\mathbb{Z}[G]} C^*(M) & \underset{\cong}{\xrightarrow{\times_G}} & C^{d_r+*}(EG_r \times_G M) \\ PD_{EG_r} \otimes PD_M \downarrow & & \downarrow PD_{EG_r \times_G M} \\ C_{-*}(EG_r) \otimes_{\mathbb{Z}[G]} C^{lf}_{m-*}(M) & \underset{\cong}{\xrightarrow{\times_G}} & C^{lf}_{m-*}(EG_r \times_G M) \end{array}$$

Proof. Let $\sigma_{EG_r} = \sum e_k \in C_{d_r}(EG_r)$ and $\sigma_M = \sum m_l \in C^{lf}_m(M)$ be the representatives of the fundamental classes of EG_r and M respectively. Let $\sigma_{EG_r,G} = \sum e_{k'} \in C_{d_r}(EG_r)$ be a chain with the property that $\sum_{g \in G} g \cdot \sigma_{EG_r,G} = \sigma_{EG_r}$ then $\sigma_{EG_r,G} \otimes \sigma_M$ is the representative of the fundamental class of $EG_r \times_G M$ in $C^{lf}_{d_r+m}(EG_r \times_G M) \cong C_{d_r}(EG_r) \otimes_{\mathbb{Z}[G]} C^{lf}_m(M)$. Choose equivariant cellular approximations to the diagonal maps: $\Delta_1 : EG_r \to EG_r^2$, $\Delta_2 : M \to M^2$ and using these maps choose $\Delta : EG_r \times_G M \to (EG_r \times_G M)^2$. Denote $\Delta_{1*}(\sigma_{EG_r,G}) = \sum_i e^1_i \otimes e^2_{d_r-i}$ and $\Delta_{2*}(\sigma_M) = \sum_j m^1_j \otimes m^2_{m-j}$ then:

$$\Delta_*(\sigma_{EG_r,G} \otimes \sigma_M) = \sum_{i,j} (-1)^{(d_r-i)j} e^1_i \otimes_G m^1_j \otimes e^2_{d_r-i} \otimes_G m^2_{m-j}.$$

We follow the image of an element $\varphi \otimes \psi \in C^{d_r+*}(EG_r) \otimes_{\mathbb{Z}[G]} C^*(M)$. If we first go down and then right we get:
$(\varphi \cap \sigma_{EG_r}) \otimes (\psi \cap \sigma_M) = \varphi \cap (\Sigma_g g \cdot \sigma_{EG_r,G}) \otimes_G \psi \cap \sigma_M$
$= \Sigma_g (\varphi \cap (g \cdot \sigma_{EG_r,G}) \otimes_G \psi \cap (g \cdot \sigma_M))$ (σ_M is invariant - $g \cdot \sigma_M = \sigma_M$)
$= \Sigma_g \left((-1)^{|\varphi|(d_r-|\varphi|)} \varphi(ge^2_{|\varphi|}) \cdot ge^1_{d_r-|\varphi|} \otimes_G (-1)^{|\psi|(m-|\psi|)} \psi(gm^2_{|\psi|}) \cdot gm^1_{m-|\psi|}\right)$
$= (-1)^{|\varphi|(d_r-|\varphi|)+|\psi|(m-|\psi|)} \left(\Sigma_g \varphi(ge^2_{|\varphi|}) \cdot \psi(gm^2_{|\psi|})\right) \cdot e^1_{d_r-|\varphi|} \otimes_G m^1_{m-|\psi|}$
$= (-1)^{|\varphi|(d_r-|\varphi|)+|\psi|(m-|\psi|)+|\varphi||\psi|} \varphi \times_G \psi(e^2_{|\varphi|} \otimes_G m^2_{|\psi|}) \cdot e^1_{d_r-|\varphi|} \otimes_G m^1_{m-|\psi|}$
$= \varphi \times_G \psi \cap \sigma_{EG_r,G} \otimes \sigma_M$
which is equal to the image if we first go right and then down. □

The composition of the following maps:

$$C^{*+d_r}(EG_r \times_G M) \to C^{*+d_r}(EG_r) \otimes_{\mathbb{Z}[G]} C^*(M) \to P^+_{-*} \otimes_{\mathbb{Z}[G]} C^*(M) \to$$
$$\to Hom_{\mathbb{Z}[G]}(P^-_*, \mathbb{Z}[G]) \otimes_{\mathbb{Z}[G]} C^*(M) \to Hom_{\mathbb{Z}[G]}(P^-_*, C^*(M))$$

induces a map $H^{k+d_r}(EG_r \times_G M) \to DH^k_G(M)$. The second map is defined using the fact that $C^{*+d_r}(EG_r)$ has an augmentation $C^{d_r}(EG_r) \to \mathbb{Z}$ given by evaluation on the top class so the map $C^{*+d_r}(EG_r) \to P^+_{-*}$ is the unique (up to homotopy) augmentation preserving map given by the universal property of P^+_* ([6] I,7.4).

Lemma 48. *The following diagram commutes:*

$$\begin{array}{ccc} H^{k+d_r}(EG_r \times_G M) & \to & DH^k_G(M) \\ \downarrow PD_{EG_r \times_G M} & & \downarrow PD_M \\ H^{lf}_{m-k}(EG_r \times_G M) & \to & H^{lf,G}_{m-k}(M) \end{array}$$

Proof. We prove that the following diagram commutes up to homotopy:

$$\begin{array}{ccccc} C^{d_r+*}(EG_r \times_G M) & \to & C^{d_r+*}(EG_r) \otimes_{\mathbb{Z}[G]} C^*(M) & \to & P^+_{-*} \otimes_{\mathbb{Z}[G]} C^*(M) \\ \downarrow PD_{EG_r \times_G M} & & PD_{EG_r} \otimes PD_M \downarrow & & PD_M \downarrow \\ C^{lf}_{m-*}(EG_r \times_G M) & \to & C_{-*}(EG_r) \otimes_{\mathbb{Z}[G]} C^{lf}_{m-*}(M) & \to & P^+_{-*} \otimes_{\mathbb{Z}[G]} C^{lf}_{m-*}(M) \end{array}$$

The left square commutes by lemma 47. The commutativity of the right square follows from the fact that the map $C^{d_r+*}(EG_r) \xrightarrow{PD_{EG_r}} C_{-*}(EG_r)$ is augmentation preserving, which is a simple check. $\square$

Proposition 49. *Θ^D_G is a natural transformation.*

Proof. Let $f : N \to M$ be a smooth equivariant map between two smooth oriented manifolds of dimension n and m resp. with a smooth and orientation preserving G action. We would like to show that the following diagram commutes:

$$\begin{array}{ccc} DSH^k_G(M) & \xrightarrow{\Theta^D_G} & DH^k_G(M) \\ f^* \downarrow & & \downarrow f^* \\ DSH^k_G(N) & \xrightarrow{\Theta^D_G} & DH^k_G(N) \end{array}$$

First case - $f : N \hookrightarrow M$ is a closed embedding:

Take an element $\alpha = [S, g] \in DSH^k_G(M)$. We can assume that g is transversal to f, thus we can find a (closed) invariant tubular neighborhood U of N with boundary ∂U ([5] VI,2.2) (also transversal to g) and a projection map $\pi_N : U \to N$ with the property that the pullback of U will be a tubular neighbourhood:

$$\pi_S : S \pitchfork U \to S \pitchfork N.$$

Let EG_r be as above, we claim that the following diagram commutes:

$$\begin{array}{ccccccc} H^{lf}_{m-k}(S/G) & \xrightarrow{(h \times_G g)_*} & H^{lf}_{m-k}(EG_r \times_G M) & \to & H^{lf,G}_{m-k}(M) & \xrightarrow{PD^{-1}_M} & DH^k_G(M) \\ \varepsilon \cdot \phi \downarrow & (1) & \downarrow \varepsilon \cdot \phi & & (2) & & \downarrow f^* \\ H^{lf}_{n-k}(S \pitchfork N/G) & \xrightarrow{(h \times_G g)_*} & H^{lf}_{n-k}(EG_r \times_G N) & \to & H^{lf,G}_{n-k}(N) & \xrightarrow{PD^{-1}_N} & DH^k_G(N) \end{array}$$

i) $h : S \to EG_r$ is the classifying map defined by the universal property of EG_r and the fact that S has the homotopy type of a CW complex of dimension $\leq m-k < r$ and has a free G action.
ii) The maps ϕ are induced by the Thom isomorphism, and $\varepsilon = (-1)^{(n-k)(m-n)}$.

More details can be found in Appendix 1 of [25].
(1) commutes by the naturality of the Thom class. To prove that (2) commutes it is enough (using lemma 48) to show that the following diagram commutes:

$$\begin{array}{ccccc} H^{lf}_{m-k}(EG_r \times_G M) & \to & H^{k+d_r}(EG_r \times_G M) & \to & DH^k_G(M) \\ \downarrow \varepsilon \cdot \phi & & \downarrow f^* & & \downarrow f^* \\ H^{lf}_{n-k}(EG_r \times_G N) & \to & H^{k+d_r}(EG_r \times_G N) & \to & DH^k_G(N) \end{array}.$$

Commutativity of the left side is proved in ([25] 7.21), the right side clearly commutes.

We follow both images of $[S/G]^{lf} \in H^{lf}_{m-k}(S/G)$ in the original diagram. By definition, the image of $[S/G]^{lf}$ in the top row is $\Theta^D_G(\alpha)$, which is mapped in the right column to $f^*(\Theta^D_G(\alpha))$. As proved in ([25] 7.22) $\varepsilon \cdot \phi([S/G]^{lf}) = [S \pitchfork N/G]^{lf}$. By definition its image in the bottom row is equal to $\Theta^D_G(f^*(\alpha))$ using the fact that $[S \pitchfork N/G, g]^{lf} = f^*(\alpha)$. Since the diagram commutes we conclude that $f^*(\Theta^D_G(\alpha)) = \Theta^D_G(f^*(\alpha))$.

The general case - $f : N \to M$ is not necessarily an embedding:

Let $i : N \hookrightarrow V^G$ be a smooth, closed, equivariant embedding in a finite dimensional representation (see [21]). f is the composition $N \xrightarrow{f\times i} M \times V^G \xrightarrow{\pi_M} M$. $f \times i$ is a closed equivariant embedding. π_M has an inverse up to G homotopy which is an embedding - $M \xrightarrow{Id\times 0} M \times V^G$ hence this follows from the previous case.

□

Lemma 50. *Θ^D_G commutes with the cross product, that is for M and N as above the following diagram commutes:*

$$\begin{array}{ccc} DSH^k_G(M) \otimes DSH^l_{G'}(N) & \xrightarrow{\Theta^D_G \otimes \Theta^D_{G'}} & DH^k_G(M) \otimes DH^l_{G'}(N) \\ \times\downarrow & & \times\downarrow \\ DSH^{k+l}_{G\times G'}(M \times N) & \xrightarrow{\Theta^D_{G\times G'}} & DH^{k+l}_{G\times G'}(M \times N) \end{array}$$

Proof. The following diagram commutes up to sign $(-1)^{ml}$ since the horizontal maps are identities and the vertical maps are equal up to that exact sign:

$$\begin{array}{ccc} DSH^k_G(M) \otimes DSH^l_{G'}(N) & \xrightarrow{PD_M \otimes PD_N} & SH^{lf,G}_{m-k}(M) \otimes SH^{lf,G'}_{n-l}(N) \\ \times\downarrow & & \downarrow\times \\ DSH^{k+l}_{G\times G'}(M \times N) & \xrightarrow{PD_{M\times N}} & SH^{lf,G\times G'}_{m+n-k-l}(M \times N) \end{array}$$

The following diagram commutes:

$$\begin{array}{ccc} SH^{lf,G}_{m-k}(M) \otimes SH^{lf,G'}_{n-l}(N) & \xrightarrow{\Phi^{lf}_G \otimes \Phi^{lf}_G} & H^{lf,G}_{m-k}(M) \otimes H^{lf,G'}_{n-l}(N) \\ \downarrow\times & & \downarrow\times \\ SH^{lf,G\times G'}_{m+n-k-l}(M \times N) & \xrightarrow{\Phi^{lf}_{G\times G'}} & H^{lf,G\times G'}_{m+n-k-l}(M \times N) \end{array}$$

since Φ^{lf}_G commutes with the cross product (as was shown before).
And last, the following diagram commutes up to sign $(-1)^{ml}$:

$$\begin{array}{ccc} H^{lf,G}_{m-k}(M) \otimes H^{lf,G'}_{n-l}(N) & \xrightarrow{PD^{-1}_M \otimes PD^{-1}_N} & DH^k_G(M) \otimes DH^l_{G'}(N) \\ \downarrow\times & & \times\downarrow \\ H^{lf,G\times G'}_{m+n-k-l}(M \times N) & \xrightarrow{PD^{-1}_{M\times N}} & DH^{k+l}_{G\times G'}(M \times N) \end{array}$$

To see that take an element $p \otimes \varphi \otimes q \otimes \psi \in P_G^+ \otimes C^*(M) \otimes P_{G'}^+ \otimes C^*(N)$. If we first go left and then down we get:
$p \otimes \varphi \otimes q \otimes \psi \mapsto p \otimes (\varphi \cap \sigma_M) \otimes q \otimes (\psi \cap \sigma_N) \mapsto (-1)^{|q|\cdot|\varphi\cap\sigma_M|} p \otimes q \otimes (\varphi \cap \sigma_M) \otimes (\psi \cap \sigma_N)$
where σ_M and σ_N are the representatives of the fundamental classes of M and N.
If we first go down and then left we get:
$p \otimes \varphi \otimes q \otimes \psi \mapsto (-1)^{|q|\cdot|\varphi|} p \otimes q \otimes \varphi \otimes \psi \mapsto (-1)^{|q|\cdot|\varphi|} p \otimes q \otimes (\varphi \otimes \psi) \cap (\sigma_M \otimes \sigma_N)$
($\sigma_M \otimes \sigma_N$ is the representative of the fundamental class of $M \times N$).
$= (-1)^{|q|\cdot|\varphi| + m\cdot|\psi|} p \otimes q \otimes (\varphi \cap \sigma_M) \otimes (\psi \cap \sigma_M)$
Comparing the signs we get that the diagram commutes up to the sign:
$(-1)^{|q|\cdot|\varphi| + m\cdot|\psi|} \cdot (-1)^{|q|\cdot|\varphi\cap\sigma_M|}$ which is equal to $(-1)^{m\cdot(|\psi|+|q|)} = (-1)^{m\cdot l}$
Combining the three diagrams we get the commutativity of the original diagram. □

Lemma 51. Θ_G^D *commutes with the coboundary operator.*

Proof. This is similar to what is done for Θ. □

Lemma 52. Θ_G^D *commutes with the restriction map.*

Proof. This follows from the fact that the maps:

$$DSH_G^k(M) \to SH_{m-k}^{lf,G}(M) \to H_{m-k}^{lf,G}(M) \to DH_G^k(M)$$

commute with the restriction in cohomology and transfer in homology. □

Altogether, we proved the following:

Theorem 53. $\Theta_G^D : DSH_G^*(M) \to DH_G^*(M)$ *is a natural isomorphism of multiplicative equivariant cohomology theories.*

3.4. **Equivariant stratifold cohomology and its dual stratifold backwards homology.**

Let G be a compact Lie group and M a smooth oriented manifold of dimension m with an orientation preserving smooth G action. Let EG_r be an increasing sequence of manifolds as we had before. Define the equivariant stratifold cohomology:

$$SH_G^k(M) = \varprojlim DSH_G^k(EG_r \times M)$$

with respect to the maps induced by the inclusions $EG_r \times M \hookrightarrow EG_{r+1} \times M$. This inverse limit stabilizes due to the following:

Lemma 54. *For r large enough the maps $DSH_G^k(EG_{r+1} \times M) \to DSH_G^k(EG_r \times M)$ are isomorphisms.*

Proof. The following diagram commutes and all the maps are isomorphisms:

$$\begin{array}{ccccc}
DSH_G^k(EG_{r+1} \times M) & \overset{\cong}{\longrightarrow} & SH^k(EG_{r+1} \times_G M) & \overset{\cong}{\longrightarrow} & H^k(EG_{r+1} \times_G M) \\
\downarrow & & \downarrow & & \downarrow\cong \\
DSH_G^k(EG_r \times M) & \overset{\cong}{\longrightarrow} & SH^k(EG_r \times_G M) & \overset{\cong}{\longrightarrow} & H^k(EG_r \times_G M)
\end{array}$$

□

$SH_G^*(M)$ has a natural structure of an Abelian group. As we saw before, one can define triples and a boundary map for a triple and prove a Mayer Vietoris theorem. We can also define the cross product. SH_G^* with the boundary operator and the product is a multiplicative equivariant cohomology theory.

The maps $DSH^k_G(M) \to DSH^k_G(M \times EG_r)$ induce a (multiplicative) natural transformation $DSH^k_G(M) \to SH^k_G(M)$. If the action on M is free this is an isomorphism since then it is equal to the composition:

$$DSH^k_G(M) \xrightarrow{\cong} SH^k(M/G) \xrightarrow{\cong} SH^k_G(M)$$

but in general it is not an isomorphism.

In a similar way, one can define a dual theory stratifold backwards homology, but we will not give the details here.

A natural isomorphism between SH^*_G and H^*_G.

Let G be a finite group, and M a smooth oriented manifold with a smooth and orientation preserving action. Choose EG_r as before, where r is large enough. Using the following diagram:

$$\begin{array}{ccccc} DSH^k_G(M) & \xrightarrow{\cong} & DH^k_G(M) & \to & H^k_G(M) \\ \downarrow & & \downarrow & & \cong\downarrow \\ DSH^k_G(M \times EG_r) & \xrightarrow{\cong} & DH^k_G(M \times EG_r) & \xrightarrow{\cong} & H^k_G(M \times EG_r) \end{array}$$

we define a natural transformation $\Theta_G : SH^k_G(M) \to H^k_G(M)$ to be the map induced by the composition (when passing to the limit):

$$DSH^k_G(M \times EG_r) \xrightarrow{\cong} DH^k_G(M \times EG_r) \xrightarrow{\cong} H^k_G(M \times EG_r) \xrightarrow{\cong} H^k_G(M)$$

All maps are natural, so the same is true for the limit. We deduce the following:

Theorem 55. *$\Theta_G : SH^*_G(M) \to H^*_G(M)$ is a natural isomorphism of multiplicative equivariant cohomology theories, and the following diagram commutes:*

$$\begin{array}{ccc} DSH^k_G(M) & \xrightarrow{\Theta^D_G} & DH^k_G(M) \\ \downarrow & & \downarrow \\ SH^k_G(M) & \xrightarrow{\Theta_G} & H^k_G(M) \end{array}$$

Remark 56. One can construct a natural isomorphism:

$$\Phi^{lf,G}_D : DSH^{lf,G}_k(X) \to DH^{lf,G}_k(X)$$

such that the following diagram commutes ([25] 6.45):

$$\begin{array}{ccc} SH^{lf,G}_k(X) & \xrightarrow{\Phi^{lf,G}} & H^{lf,G}_k(X) \\ \downarrow & & \downarrow \\ DSH^{lf,G}_k(X) & \xrightarrow{\Phi^{lf,G}_D} & DH^{lf,G}_k(X) \end{array}$$

and also that $\Theta_G : SH^*_G(M) \to H^*_G(M)$ is equal to the composition:

$$SH^k_G(X) \to DSH^{lf,G}_{m-k}(X) \to DH^{lf,G}_{m-k}(X) \to H^k_G(X)$$

3.5. **Stratifold Tate cohomology and its dual stratifold Tate homology.**

Stratifold Tate cohomology.

Given a natural transformation between two bordism theories one can define a "relative term" using maps from manifolds with boundary, in such a way that it will give a long exact sequence. In our case, SH_G^* was not defined as a bordism theory, just because we deal with manifolds and there is no model for EG in this world. This make thing a bit more technical, but there is no essential difference. We now define this "relative term" which we call stratifold Tate cohomology. As we will see, it is isomorphic to Tate cohomology when G is finite. Due to technical difficulty, we could not prove that this isomorphism is natural, but we expect that it is.

Let G be a compact Lie group and M a smooth oriented manifold of dimension m with an orientation preserving smooth G action. For every r we define $\widehat{SH}_G^k(M)_r$ to be $\{(T,S,g,i_\partial)\}/\sim$ where:

- T is an oriented stratifold of dimension $m+d_r-k$ (where d_r is the dimension of EG_r) with boundary together with an orientation preserving, smooth and free G action.
- S is an oriented stratifold of dimension $m-k-1$ together with an orientation preserving, smooth and free G action.
- $i_\partial : EG_r \times S \to \partial T$ is an orientation preserving, equivariant isomorphism.
- $g : T \to EG_r \times M$ is a proper equivariant smooth map and the composition $g \circ i_\partial : EG_r \times S \to EG_r \times M$ is of the form $Id \times \tilde{g}$ for some equivariant smooth map $\tilde{g} : S \to M$.

The bordism relation is defined the following way: We say that (T,S,g,i_∂) is bordant to (T',S',g',i'_∂) if the following conditions hold:

- There exists a bordism $(B, S \amalg -S', h)$ between $\tilde{g} : S \to M$ and $\tilde{g}' : S' \to M$ (i.e., $[S,\tilde{g}] = [S',\tilde{g}']$ as elements in $DSH_G^{k+1}(M)$).
  Thus $(EG_r \times B, EG_r \times (S \amalg S'), Id \times h)$ is a bordism between $Id \times \tilde{g} : EG_r \times S \to EG_r \times M$ and $Id \times \tilde{g}' : EG_r \times S' \to EG_r \times M$. (Note that B might be non empty even if $S = S' = \emptyset$).
- By gluing $(EG_r \times B, EG_r \times S \amalg EG_r \times S')$ to $(T,\partial T) + (T',\partial T')$ along $i_\partial + i'_\partial$ we obtain an oriented stratifold of dimension $m+d_r-k$ together with a G action which is orientation preserving, smooth and free, which we denote by $\tilde{B}$. We require that the element $[g+g'+Id \times h : \tilde{B} \to EG_r \times M] \in DSH_G^k(EG_r \times M)$ will be the zero element.

Proposition 57. *The bordism relation is an equivalence relation.*

Proof. **The relation is reflexive:** Given an element (T,S,g,i_∂) one can give $T \times I$ the structure of a stratifold with boundary such that its boundary will be equal to $T \times \{0\} \cup \partial T \times I \cup T \times \{1\}$ by a similar procedure to the one appears in [16] in appendix A. This implies that (T,S,g,i_∂) is equivalent to itself.
The relation is symmetric: This is clear.
The relation is transitive: In order to prove this we have to know how to glue two stratifolds along a part of their boundary. This is proved in [16] in appendix A. □

The maps $i : EG_r \to EG_{r+1}$ induce maps $i^* : \widehat{SH}_G^k(M)_{r+1} \to \widehat{SH}_G^k(M)_r$ by transversal pullback. Notice that the pullback of $Id \times \widetilde{g} : EG_{r+1} \times S \to EG_{r+1} \times M$ along the map $EG_r \times M \to EG_{r+1} \times M$ is equal to $Id \times \widetilde{g} : EG_r \times S \to EG_r \times M$, hence the map

is well defined. Define the stratifold Tate cohomology to be $\varprojlim \widehat{SH}^k_G(M)_r$. This limit stabilizes due to the following:

Lemma 58. *The maps $i^* : \widehat{SH}^k_G(M)_{r+1} \to \widehat{SH}^k_G(M)_r$ are isomorphisms for r large enough.*

Proof. This follows from the following diagram and the five lemma :

$$\begin{array}{ccccccccc}
DSH^k_G(M) & \to & DSH^k_G(EG_{r+1}\times M) & \to & \widehat{SH}^k_G(M)_{r+1} & \to & DSH^{k+1}_G(M) & \to & DSH^{k+1}_G(EG_{r+1}\times M) \\
\downarrow Id & & \downarrow i^* & & \downarrow i^* & & \downarrow Id & & \downarrow i^* \\
DSH^k_G(M) & \to & DSH^k_G(EG_r\times M) & \to & \widehat{SH}^k_G(M)_r & \to & DSH^{k+1}_G(M) & \to & DSH^{k+1}_G(EG_r\times M)
\end{array}$$

□

Remark 59. The exactness of the rows and the definitions of the natural transformations are explained below.

$\widehat{SH}^*_G(M)$ has a natural structure of an Abelian group. As we saw before, one can define triples and a boundary map for a triple and prove a Mayer-Vietoris theorem. $\widehat{SH}^*_G$ with the boundary operator is an equivariant cohomology theory.

There are natural transformations:

- $DSH^*_G \to SH^*_G$ given by $[S \to M] \mapsto [EG_r \times S \to EG_r \times M]$ which is induced by π^* where $\pi : EG_r \times M \to M$ is the projection.
- $SH^*_G \to \widehat{SH}^*_G$ given by $[(T, g)] \mapsto [(T, \emptyset, g, \emptyset)]$.
- $\widehat{SH}^*_G \to DSH^{*+1}_G$ given by $[(T, S, g, i_\partial)] \mapsto [(S, \tilde{g})]$.

We have the following:

Theorem 60. *The following is a long exact sequence:*
$... \to DSH^*_G \to SH^*_G \to \widehat{SH}^*_G \to DSH^{*+1}_G \to ...$

Proof. This follows easily from the definition of the bordism relation in $\widehat{SH}^*_G$. To prove exactness in $\widehat{SH}^*_G$ one might use the following fact that is trivial from the definition of the bordism relation: Let (T, S, g, i_∂) be as before and assume that the map $\tilde{g} : S \to M$ bounds the map $\bar{g} : (S', S) \to M$. Denote by $(T', \emptyset, \hat{g}, \emptyset) = (T, \hat{g})$ the element we get by gluing (T, S, g, i_∂) and $(EG_r \times S', S, \hat{g}, i)$ where i is the inclusion of $EG_r \times S$ in $EG_r \times S'$. Then (T, S, g, i_∂) is bordant to $(T', \emptyset, \hat{g}, \emptyset) = (T, \hat{g})$. □

Proposition 61. *$\widehat{SH}^*_G(M)$ vanish if and only if G acts freely on M.*

Proof. As noted before, for free action we have $DSH^*_G(M) \xrightarrow{\cong} SH^*_G(M)$, so by exactness we conclude that Tate cohomology vanishes.

If the action is not free we can find an orbit of the form G/H for some non trivial closed subgroup $H \leq G$. By ([24], 2) the map $f^* : H^k(BG) \to H^k(BH)$ (or equivalently the map $f^* : SH^k_G(pt) \to SH^k_G(G/H)$) is non zero for an infinite number of values of k. Since this map factors through $SH^k_G(M)$ the same is true for these groups. This implies the proposition since, by exactness, the map $SH^k_G(M) \to \widehat{SH}^k_G(M)$ is an isomorphism above the dimension of M. □

The following is proved in a similar way to corollary 21:

Corollary 62. *Let M be a smooth oriented manifold with an orientation preserving smooth G action. An element $\alpha \in DSH^k_G(M)$ is in the kernel of the map $DSH^k_G(M) \to SH^k_G(M)$ if and only if it is in the kernel of the map $f^* : DSH^k_G(M) \to DSH^k_G(N)$ for every equivariant map $f : N \to M$ where N is a smooth oriented manifold with an orientation preserving smooth free G action.*

For $M \times G$ we have:

$$DSH^k_G(M \times G) \cong SH^k(M \times_G G) \cong SH^k(M)$$

Under this identification, the image of an element $[S \to M] \in DSH^k_G(M)$ is $[S \to M] \in SH^k(M)$, giving us the following easy criterion:

Corollary 63. *Let M be as above, and $[S \to M] \in DSH^k_G(M)$. If $[S \to M]$ is non trivial as an element in $SH^k(M)$ then it is not in the kernel of the map $DSH^k_G(M) \to SH^k_G(M)$.*

Stratifold Tate homology.

Let G be a fixed compact Lie group, we introduce stratifold Tate homology and denote it by $\widehat{SH}^G_*$. It is defined on the category of finite dimensional $G-CW$ complexes and equivariant cellular maps between them. The definition is like for stratifold Tate cohomology so we will not repeat it. Induced maps are defined by composition.

$\widehat{SH}^G_*(X)$ has a natural structure of an Abelian group. Like before, one can define triples and a boundary map for a triple and we will have the Mayer-Vietoris long exact sequence.

As before, there are obvious forms of equivariant duality we call Poincaré duality:

Theorem 64. *Let M be a closed oriented smooth manifold of dimension m with a smooth and orientation preserving G action then there is an isomorphism:*

$$PD_M : \widehat{SH}^k_G(M) \to \widehat{SH}^G_{m-k}(M)$$

In case M is not compact we have the following version:

$$PD_M : \widehat{SH}^k_G(M) \to \widehat{SH}^{lf,G}_{m-k}(M).$$

There are natural transformations:

- $SH^G_* \to DSH^G_*$ given by $[S \to X] \mapsto [EG_r \times S \to EG_r \times X]$.
- $DSH^G_* \to \widehat{SH}^G_*$ given by $[T, g] \mapsto [T, \emptyset, g, \emptyset]$.
- $\widehat{SH}^G_* \to SH^G_{k-1}$ given by $[T, S, g, i_\partial] \mapsto [S, \tilde{g}]$.

We have the following (the proof is the same as for cohomology):

Proposition 65. *The following is a long exact sequence:*

$$... \to SH^G_* \to DSH^G_* \to \widehat{SH}^G_* \to SH^G_{*-1} \to ...$$

A natural isomorphism between $\widehat{SH}^{lf,G}_*$ and $\hat{H}^{lf,G}_{*-1}$.

We start with a short discussion about the algebraic mapping cone, details can be found in Weibel [27].

The algebraic mapping cone: Let A and B be two chain complexes and $f : A \to B$ be a chain map. The algebraic mapping cone (or just mapping cone) of

f, denoted by C_f, is the chain complex $C_{f,k} = B_{k+1} \oplus A_k$ with the differential given by $\partial(b,a) = (f(a) - \partial b, \partial a)$. Here are some properties of the mapping cone.

(1) The maps $B[-1] \to C_f$, given by $b \mapsto (b,0)$, and $C_f \to A$, given by $(b,a) \mapsto a$, are chain maps and the following is an exact sequence:

$$0 \to B[-1] \to C_f \to A \to 0$$

The connecting homomorphism in the induced long exact sequence in homology is equal to f_*.

(2) The mapping cone is functorial in the category of chain complexes and chain maps.

(3) Given a square of chain complexes which commutes up to homotopy s:

$$(*) \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ h_A \downarrow & & \downarrow h_B \\ A' & \xrightarrow{f'} & B' \end{array}$$

then the map $h_C : C_f \to C_{f'}$ given by $h_C(b,a) = (h_B(b) + s(a), h_A(a))$ is a chain map and the following diagram commutes:

$$\begin{array}{ccccc} B[-1] & \to & C_f & \to & A \\ \downarrow h_B & & \downarrow h_C & & h_A \downarrow \\ B'[-1] & \to & C_{f'} & \to & A' \end{array}$$

inducing a map between the long exact sequences mentioned in 1).

Remark 66. h_C depends on the choice of s, even after passing to homology. Replacing any of the maps by another chain map which is chain homotopic to it might change the map induced in homology by h_C. These facts make the mapping cone non functorial in the homotopy category of chain complexes, which leads to a great deal of trouble. We will have to overcome this trouble again and again, since most of our maps are defined only up to chain homotopy, depending on choices that we made.

A fundamental class for a pair $(T, EG_r \times S)$.

Remark. For brevity, for an oriented stratifold P (with boundary) of dimension j denote by $\sigma_P \in C_j^{lf}(P)$ the representatives of its fundamental class.

Let G be a finite group, and $(T, EG_r \times S)$ as above. σ_{EG_r} is unique and thus invariant since we are using cellular chains, thus we have a map $\mathbb{Z} \to C_{d_r}(EG_r)$. Define a degree d_r chain map: $\rho : C_*(EG_r) \to C_{*+d_r}(EG_r)$ the following way:
Let $\rho : C_0(EG_r) \to C_{d_r}(EG_r)$ be the composition $C_0(EG_r) \to \mathbb{Z} \to C_{d_r}(EG_r)$, else let $\rho : C_k(EG_r) \to C_{k+d_r}(EG_r)$ be zero map.

Look at the map $\bar{\rho} : C_*^{lf}(S/G) \to C_{*+d_r}^{lf}(T/G)$ defined as the composition:

$$C_*^{lf}(S/G) \to C_*(EG_r) \otimes C_*^{lf}(S) \xrightarrow{\rho \otimes Id} C_{*+d_r}(EG_r) \otimes C_*^{lf}(S) \xrightarrow{i_{\partial *}} C_{*+d_r}^{lf}(T/G)$$

Denote the mapping cone of this map by $C_*^{lf}(T,S)$.

Define the fundamental class $\widehat{[T,S]}^{lf}$ to be the class of $(\sigma_{T/G}, \sigma_{S/G}) \in C_{k-1}^{lf}(T,S)$. This is a cycle since:

$$\partial(\sigma_{T/G}, \sigma_{S/G}) = (i_{\partial *}(\sigma_{EG_r \times_G S}) - \partial \sigma_{T/G}, \partial \sigma_{S/G}) = (0,0)$$

Let $(T, EG_r \times S)$ be as above and let (T'', T') be a null bordism, that is an oriented stratifold of dimension $k+d_r+1$ with boundary, together with a free and orientation preserving G action, such that its boundary is obtained by gluing $(T, EG_r \times S)$ with an element of the form $(S' \times EG_r, S \times EG_r)$ along the boundary.

Lemma 67. *The inclusion induces a commutative square:*

$$\begin{array}{ccc} C_*(S/G) & \to & C_*(S'/G) \\ \downarrow & & \downarrow \\ C_{*+d_r}(T/G) & \to & C_{*+d_r}(T''/G) \end{array}$$

which induces a map between the mapping cones, mapping $\widehat{[T,S]}^{lf}$ *to a boundary.*

Proof. This follows from the fact that:
$\partial(-\sigma_{T''/G}, -\sigma_{S'/G}) = (\partial\sigma_{T''/G} - \sigma_{S' \times_G EG_r}, -\partial\sigma_{S'/G}) = (\sigma_{T/G}, \sigma_{S/G})$. $\square$

Construction of the natural isomorphism.

Let X be a locally finite, finite dimensional $G-CW$ complex where G is a finite group. We would like to construct a natural isomorphism

$$\widehat{\Phi}^{lf,G} : \widehat{SH}_k^{lf,G}(X) \to \hat{H}_{k-1}^{lf,G}(X).$$

Choose $r > dim(X) - k + 1$. Let α be an element in $\widehat{SH}_k^{lf,G}(X)_r$ and choose a representative (T, S, g, i_∂). Choose augmentation preserving chain maps:

$$l^+ : C_*(EG_r) \to P_*^+ \ \textit{and}\ \ l^- : C_{*+d_r}(EG_r) \to P_*^-$$

(the second map is defined only for $* > -r$ but this is not a problem since we are only interested in r large). We have the following commutative diagram:

$$\begin{array}{ccccc} C_*(EG_r) & \to & \mathbb{Z} & \to & C_{*+d_r}(EG_r) \\ \downarrow l^+ & & \downarrow Id & & \downarrow l^- \\ P_*^+ & \to & \mathbb{Z} & \to & P_*^- \end{array}$$

Look at the following diagram (h_* is induced by the map $S \to EG_r \times S$):

$$\begin{array}{ccccccc} C_*^{lf}(S/G) & \xrightarrow{h_*} & C_*(EG_r) \otimes C_*^{lf}(S) & \xrightarrow{Id \otimes g_*} & C_*(EG_r) \otimes C_*^{lf}(X) & \xrightarrow{l^+ \otimes Id} & P_*^+ \otimes C_*^{lf}(X) \\ & & \downarrow \rho \otimes Id & & \downarrow \rho \otimes Id & & \downarrow \\ & & C_{*+d_r}(EG_r) \otimes C_*^{lf}(S) & \xrightarrow{Id \otimes g_*} & C_{*+d_r}(EG_r) \otimes C_*^{lf}(X) & \xrightarrow{l^- \otimes Id} & P_*^- \otimes C_*^{lf}(X) \end{array}$$

where all tensor products are over $\mathbb{Z}[G]$. The map $l^- \otimes Id$ is defined only for $k > dim(X) - r$, this is the reason we have to choose $r > dim(X) - k + 1$.

The map $EG_r \times_G S \to EG_r \times_G X$ factors through T/G thus we get the following commutative diagram:

$$\begin{array}{ccccc} C_*^{lf}(S/G) & \xrightarrow{g_*} & C_*(EG_r) \otimes C_*^{lf}(X) & \xrightarrow{l^+ \otimes Id} & P_*^+ \otimes C_*^{lf}(X) \\ \bar{\rho} \downarrow & & \downarrow \rho \otimes Id & & \downarrow \\ C_{*+d_r}^{lf}(T/G) & \xrightarrow{g_*} & C_{*+d_r}(EG_r) \otimes C_*^{lf}(X) & \xrightarrow{l^- \otimes Id} & P_*^- \otimes C_*^{lf}(X) \end{array}$$

This induces a map between the mapping cones. $C_{\bar{\rho}}$ was denoted by $C_*^{lf}(T, S)$, and the mapping cone of the map $P_*^+ \otimes C_*^{lf}(X) \to P_*^- \otimes C_*^{lf}(X)$ is naturally isomorphic to $P_* \otimes C_*^{lf}(X)$. We get a map $g_* : C_*^{lf}(T, S) \to P_* \otimes C_*^{lf}(X)$. Define $\widehat{\Phi}^{lf,G}(\alpha) = g_*(\widehat{[T,S]}^{lf})$. We would like to show that $\widehat{\Phi}^{lf,G}(\alpha)$ is independent of all the choices made:

(1) The choice of the representative (T, S, g, i_∂) and the choices of the cellular approximations $C_*^{lf}(S/G) \xrightarrow{g_*} C_*(EG_r) \otimes C_*^{lf}(X)$ and $C_*^{lf}(T/G) \xrightarrow{g_*} C_*(EG_r) \otimes C_*^{lf}(X)$:
This follows from lemma 67.
(2) The choices of the maps $C_*(EG_r) \xrightarrow{l^+} P_*^+$ and $C_{*+d_r}(EG_r) \xrightarrow{l^-} P_*^-$:
We choose those maps once and for all.
(3) The choice of the map $h_* : C_*^{lf}(S/G) \to C_*(EG_r) \otimes C_*^{lf}(S)$:
Let $h_*^1, h_*^2 : C_*^{lf}(S/G) \to C_*(EG_r) \otimes C_*^{lf}(S)$ be two cellular approximations, then there is a chain homotopy between them, denote it by s. A simple check (using the fact that $\rho \otimes Id\left(s(\sigma_{S/G})\right) = 0$) shows that the difference between both images of $\widehat{[T,S]}^{lf}$ is equal to $\partial\left(0, Id \otimes g_*\left(s(\sigma_{S/G})\right)\right)$ which is a boundary.
(4) The choice of r:
Let $[T', S', g', i'_\partial]$ be an element in $\widehat{SH}_k^{lf,G}(X)_{r+1}$ and $[T, S, g, i_\partial]$ an element in $\widehat{SH}_k^{lf,G}(X)_r$ such that $[T, S, g, i_\partial] = i^*([T', S', g', i'_\partial])$. We would like to show that $\widehat{\Phi}^{lf,G}([T, S, g, i_\partial]) = \widehat{\Phi}^{lf,G}([T', S', g', i'_\partial])$. We may assume that (T, S, g, i_∂) is given as the transversal intersection of $(T', S', g', i'_\partial)$ with $EG_r \times X$ (in the sense mentioned before) so $S' = S$. $\widehat{\Phi}^{lf,G}([T', S, g', i'_\partial])$ is the class of the image of $(\sigma_{T'/G}, \sigma_{S/G})$, and $\widehat{\Phi}^{lf,G}([T, S, g, i_\partial])$ is the class of the image of $(\sigma_{T/G}, \sigma_{S/G})$, we would like to show that their images differ by a boundary.
The first thing we notice is that by choosing the maps $l^+ : C_*(EG_r) \to P_*^+$ and $l^+ : C_*(EG_{r+1}) \to P_*^+$ compatible with the map $C_*(EG_r) \to C_*(EG_{r+1})$ the images of $\sigma_{S/G}$ in $P_*^+ \otimes C_*^{lf}(X)$ will be equal. We can do it by defining P_*^+ to be the colimit of $C_*(EG_i)$ and the maps l^+ to be the limit maps. Therefore, it is enough to show the following:

Lemma 68. *Denote by γ and γ' the images of $\sigma_{T/G}$ and $\sigma_{T'/G}$ resp. in $P_*^- \otimes_{\mathbb{Z}[G]} C_*^{lf}(X)$ then $\gamma - \gamma'$ is a boundary.*

The proof is rather technical, the interested reader can find it in ([25] 6.60).

Theorem 69. *The following diagram commutes:*

$$\begin{array}{ccccccc}
SH_*^{lf,G} & \to & DSH_*^{lf,G} & \to & \widehat{SH}_*^{lf,G} & \to & SH_{*-1}^{lf,G} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_*^{lf,G} & \to & DH_*^{lf,G} & \to & \hat{H}_{*-1}^{lf,G} & \to & H_{*-1}^{lf,G}
\end{array}$$

Proof. This is clear from the construction of the natural transformation. □

Corollary 70. $\widehat{\Phi}^{lf,G} : \widehat{SH}_k^{lf,G} \to \hat{H}_{k-1}^{lf,G}$ *is a natural isomorphism.*

Proof. Naturality is clear. The fact that it is an isomorphism follows from the fact that we know that $SH_*^{lf,G} \to H_*^{lf,G}$ and $DSH_*^{lf,G} \to DH_*^{lf,G}$ are natural isomorphisms and by the 5 lemma. □

An isomorphism between $\widehat{SH}_G^*$ and $\hat{H}_G^*$.

Let G be a fixed finite group and M a smooth oriented manifold of dimension m with a smooth and orientation preserving G action. The composition

$$\widehat{SH}_G^k(M) \to \widehat{SH}_{m-k}^{lf,G}(M) \to \hat{H}_{m-k-1}^{lf,G}(M) \to \hat{H}_G^k(M)$$

is an isomorphism of groups for all oriented manifolds, denote it by $\widehat{\Theta}_G$. We do not prove that it is natural. Trying to prove it in a way similar to what we had before runs into some technical problems, nevertheless it seems like it is possible.

Corollary 71. *The following diagram commutes:*

$$\begin{array}{ccccccc} DSH_G^* & \to & SH_G^* & \to & \widehat{SH}_G^* & \to & DSH_G^{*+1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ DH_G^* & \to & H_G^* & \to & \hat{H}_G^* & \to & DH_G^{*+1} \end{array}$$

Proof. This follows from the analog diagram for homology. □

3.6. **Examples.**

Our first example is computational. Here we would like to demonstrate how such computations can be carried out (for this reason we don't give the easiest proof). The second example is a large class of torus actions, where the duality map is trivial.

Weighted circle action on a sphere.

Consider $S^{2n-1} \subseteq \mathbb{C}^n$, for integers $k_1 ... k_n$ we define an action of S^1 by:

$$\alpha \cdot (x_1 .. x_n) = (\alpha^{k_1} x_1, ..., \alpha^{k_n} x_n).$$

Denote $K = \Pi k_i$ then:

Theorem 72. $\widehat{SH}_{S^1}^k(S^{2n-1}) = \begin{cases} \mathbb{Z}/K & k\, even \\ 0 & else \end{cases}$

Proof. Let S_f^{2n-1} be the sphere with the standard (free) action. Look at the map

$$g : S_f^{2n-1} \to S^{2n-1}$$

given by

$$g(x_1, \ldots, x_n) = (x_1^{k_1}, \ldots, x_n^{k_n}) / \left\| (x_1^{k_1}, \ldots, x_n^{k_n}) \right\|$$

By comparing the Serre spectral sequence for equivariant cohomology we deduce that:

$$SH_{S^1}^k(S^{2n-1}) = H_{S^1}^k(S^{2n-1}) = \left\{ \begin{array}{ll} \mathbb{Z} & 0 \leq k < 2n \; and\, even \\ \mathbb{Z}/K & 2n \leq k \; and\, even \\ 0 & else \end{array} \right. ,$$

A similar computation can be done for equivariant homology. Then, using equivariant Poincaré duality we get:

$$DSH_{S^1}^k(S^{2n-1}) \cong SH_{2n-1-k}^{S^1}(S^{2n-1}) \cong H_{2n-2-k}^{S^1}(S^{2n-1}) = \left\{ \begin{array}{ll} \mathbb{Z} & 0 \leq k < 2n \; and\, even \\ \mathbb{Z}/K & k < 0 \; and\, odd \\ 0 & else \end{array} \right. .$$

The only non vanishing Poincaré duality maps can occur in even degrees

$$DSH_{S^1}^{2k}(S^{2n-1}) \to SH_{S^1}^{2k}(S^{2n-1})$$

when $0 \leq k < n$. In these cases both groups are infinite cyclic. Consider the following commutative square, where all groups are infinite cyclic:

$$\begin{array}{ccc} DSH^{2k}_{S^1}(S^{2n-1}) & \rightarrow & SH^{2k}_{S^1}(S^{2n-1}) \\ \downarrow & & \downarrow \\ DSH^{2k}_{S^1}(S^{2n-1}_f) & \rightarrow & SH^{2k}_{S^1}(S^{2n-1}_f) \end{array}$$

The right vertical map is an isomorphism by the spectral sequence, and the bottom map is an isomorphism since the action is free. It is enough to compute the left vertical map. Since the (umkehr) map $DSH^{2k}_{S^1}(S^{2n-1}_f) \rightarrow DSH^{2k}_{S^1}(S^{2n-1})$ is an isomorphism, it is enough to compute the composition.

Take a generator α for $DSH^{2k}_{S^1}(S^{2n-1})$, it is represented by the map:

$$h : S^{2n-2k-1}_f \rightarrow S^{2n-1}$$

given by

$$h : (x_1, \ldots, x_{n-k}) = (x_1^{k_1}, \ldots, x_{n-k}^{k_{n-k}}, 0, \cdots, 0) / \left\| (x_1^{k_1}, \ldots, x_{n-k}^{k_{n-k}}, 0, \cdots, 0) \right\|$$

The image of α under the composition is equal to the image of

$$[g : S^{2n-1}_f \rightarrow S^{2n-1}] \in DSH^0_{S^1}(S^{2n-1})$$

under the composition:

$$DSH^0_{S^1}(S^{2n-1}) \rightarrow DSH^0_{S^1}(S^{2n-2k-1}_f) \rightarrow DSH^{2k}_{S^1}(S^{2n-1})$$

$DSH^0_{S^1}(S^{2n-1}) \rightarrow DSH^0_{S^1}(S^{2n-2k-1}_f)$ is given by multiplication by K. To show that look at the following commutative diagram

$$\begin{array}{ccc} DSH^0_{S^1}(S^{2n-1}) & \rightarrow & DSH^0_{S^1}(S^{2n-2k-1}_f) \\ \downarrow & & \downarrow \\ SH^0(S^{2n-1}) & \rightarrow & SH^0(S^{2n-2k-1}_f) \end{array}$$

where the bottom and the right map are isomorphisms and the left one is given by multiplication by K.

The map $DSH^0_{S^1}(S^{2n-2k-1}_f) \rightarrow DSH^{2k}_{S^1}(S^{2n-1})$, which is an umkehr map, is an isomorphism, again by the spectral sequence. It follows that

$$DSH^{2k}_{S^1}(S^{2n-1}_f) \rightarrow DSH^{2k}_{S^1}(S^{2n-1})$$

is given by multiplication by K. Using the long exact sequence, the theorem follows. □

Torus action with injective localization map.

We saw before that when the action is free the Tate groups vanish and the duality map is an isomorphism. We now show a large class of examples where the duality map is always trivial. This includes, among other examples, Hamiltonian torus action on symplectic manifolds with isolated fixed points, and GKM manifold.

Let M be a smooth manifold with an orientation preserving smooth G action, and F its fixed point set. Suppose that the following localization map is injective:

$$H^k_G(M) \hookrightarrow H^k_G(F)$$

When $k > dim(F) - dim(G)$ the groups $DSH^k_G(F)$ vanish so $SH^k_G(F) \rightarrow \widehat{SH}^k_G(F)$ is an isomorphism. Then from the following diagram:

$$\begin{array}{ccc} SH^k_G(M) & \to & \widehat{SH}^k_G(M) \\ \downarrow & & \downarrow \\ SH^k_G(F) & \to & \widehat{SH}^k_G(F) \end{array}$$

we conclude the map $SH^k_G(M) \to \widehat{SH}^k_G(M)$ is injective.

This situation occurs, for example in the following setting:

Theorem 73. *(Borel) Let X be a locally finite $T = (S^1)^k$ space such that $H^*_T(X)$ is free over $H^*(BT)$ then the localization map (with integral coefficients) is injective.*

A more concrete case where the above occurs is:

Theorem 74. [26] *Let a torus T act on a symplectic compact connected manifold (M, ω) in a Hamiltonian fashion. If the fixed points are isolated then the localization map (with integral coefficients) is injective.*

Appendix - Locally finite homology

In order to talk about Poincaré duality for non compact manifolds we discuss locally finite homology. This is an extract of what appears in [25], most details were omitted.

Definition 75. Let X be a space, define the locally finite chain complex $S^{lf}_k(X)$ to be the set of all formal sums of singular k simplices $\Sigma_{\sigma \in I} n_\sigma \sigma$ such that for every $x \in X$ there is an open neighborhood U such that $\{\sigma \in I | n_\sigma \neq 0 \, and \, |\sigma| \cap U \neq \emptyset\}$ is finite where $|\sigma|$ is the image of σ. $S^{lf}_*(X)$ is a chain complex. Its homology is called the locally finite homology of X and is denoted by $H^{lf}_*(X)$.

Remark. For locally compact spaces it is equivalent to the condition that every compact subset meets only finitely many simplices.

If $f : X \to Y$ is a continuous map than the image of a locally finite chain need not be locally finite as one can see in the example where X is infinite and discrete and Y is a point. It is not difficult to see that a proper map (that is, a map for which the preimage of every compact subspace is compact) $f : X \to Y$ where Y is a locally compact space induces a chain map $f_* : S^{lf}_k(X) \to S^{lf}_k(Y)$ and therefore a map $f_* : H^{lf}_k(X) \to H^{lf}_k(Y)$. If f is properly homotopic to g then $f_* = g_*$. Therefore, locally finite homology is a proper homotopy invariant.

Here are some properties of locally finite homology:

(1) For a closed subset $A \subseteq X$ one can define relative groups, just as in singular homology, and get the long exact sequence of a pair.
(2) Locally finite homology fulfills the axioms of a homology theory on the category of locally compact Hausdorff spaces and proper maps. All the proofs are standard, except maybe for excision where we refer to [22] 7.1. The proof is the same as for singular homology.
(3) For a disjoint union $\amalg X_\alpha$ we have isomorphisms $S^{lf}_*(\amalg X_\alpha) \to \Pi_\alpha S^{lf}_*(X_\alpha)$ and $H^{lf}_*(\amalg X_\alpha) \to \Pi_\alpha H^{lf}_*(X_\alpha)$.
(4) For every space X there is a chain map $S_*(X) \to S^{lf}_*(X)$ which induces a map in homology $H_*(X) \to H^{lf}_*(X)$. For a compact space X those maps are isomorphisms.

(5) If M is a connected oriented manifold of dimension m then $H_m^{lf}(M)$ is infinite cyclic with generator which we denote by $[M]^{lf}$ such that for every $x \in X$ we have $[M]^{lf} \mapsto [M, M/\{x\}]$. Moreover, if $(M, \partial M)$ is a connected oriented manifold of dimension m then $H_m^{lf}(M, \partial M)$ is infinite cyclic with generator which we denote by $[M, \partial M]^{lf}$ and $\partial[M, \partial M]^{lf} = [\partial M]^{lf}$.

For a discussion about locally finite homology the reader is referred to [14], [19] and [22].

Locally finite homology for CW complexes.

A CW complex X is called locally finite if the set of closed cells is locally finite, that is every point has a neighborhood which meets only finitely many closed cells. A CW complex is locally finite if and only if it is locally compact.

There is also a cellular version of locally finite homology. Let X be a CW complex we define $C_k^{lf}(X) = H_k^{lf}(X_k, X_{k-1})$ with the differential coming from the long exact sequence for the triple (X_k, X_{k-1}, X_{k-2}). For locally finite CW complexes we have by the properties above:

$H_k^{lf}(X_k, X_{k-1}) \cong H_k^{lf}(\amalg_I D^k, \amalg_I S^{k-1}) \cong \Pi_I H_k^{lf}(D^k, S^{k-1}) \cong \Pi_I \mathbb{Z}$

where I is the set of k cells of X.

In general the locally finite cellular homology and the locally finite singular homology need not agree. For a certain class of CW complexes they do:

Definition 76. A CW complex X is called strongly locally finite if it is the union of finite subcomplexes such that every point in X has a neighborhood which meets only finitely many of them.

Clearly, a strongly locally finite CW complex is locally finite but a locally finite CW complex needs not be strongly locally finite. An example for this is the space $X = e^0 \cup e^1 \cup e^2 ...$ where we attach each k-cell e^k to $e^0 \cup e^1 \cup e^2 ... \cup e^{k-1}$ by collapsing its boundary to a point in the interior of e^{k-1}. X is not strongly locally finite since e^0 is contained in any subcomplex (see [9] 1.8).

We have the following propositions regarding strongly locally finite CW complexes:

Proposition 77. *([9] 1.4) Every locally finite, finite dimensional CW complex is strongly locally finite.*

And the following:

Proposition 78. *([14] 4.7) For a strongly locally finite CW complex X the singular chain $S_*^{lf}(X)$ and the cellular chain $C_*^{lf}(X)$ are homology equivalent so $H_*^{lf}(X) = H_*(S_*^{lf}(X)) = H_*(C_*^{lf}(X))$.*

In [10] there is a full treatment of the cellular version.

Poincaré duality for non compact manifolds.

Poincaré duality is a deep theorem about manifolds. In its most common version it states that a closed oriented manifold of dimension m has a fundamental class $[M] \in H_m(M)$ and there is an isomorphism $PD_M : H^k(M) \to H_{m-k}(M)$ given by $\varphi \mapsto \varphi \cap [M]$. There is a similar result for smooth manifolds in locally finite homology ([19] 3.1):

Theorem 79. *Let M be a smooth oriented manifold of dimension m, not necessarily compact, then M has a fundamental class $[M]^{lf} \in H_m^{lf}(M)$ and there is an isomorphism $PD_M : H^k(M) \to H_{m-k}^{lf}(M)$ given by $\varphi \mapsto \varphi \cap [M]^{lf}$.*

The results from part 1 remain true in the locally finite setting, in particular we have the equivariant Poincaré duality.

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